Boundary Value Problems in Some Ramified Domains with a Fractal Boundary: Analysis and Numerical Methods. Part II: Non homogeneous Neumann Problems.

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Boundary Value Problems in Some Ramified Domains with a Fractal Boundary: Analysis and Numerical Methods. Part II: Non homogeneous Neumann Problems

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Abstract

This paper is devoted to numerical methods for solving Poisson problems in self-similar ramified domains of $\mathbb{R}^2$ with a fractal boundary. It is proved that a sequence of solutions to some nonhomogeneous Neumann problems posed on domains obtained by interrupting the fractal construction after a finite number of generations, converges to the solution of a Neumann problem posed in the whole domain. To define the Neumann problem on the infinitely ramified domain and for proving the above mentioned convergence, extension and trace results are given. Then, a method for computing the solution is proposed an analyzed. In particular, it is shown that the small scales of the Neumann data are damped exponentially fast away from the boundary. A self similar finite element method is developed and tested.

1 Introduction

In this paper, we deal with the numerical simulation of diffusion phenomena in a self-similar ramified domain of $\mathbb{R}^2$ with a fractal boundary. This work was motivated by a wider and very challenging project aiming at simulating the diffusion of medical sprays in the lungs. Our ambitions here are more modest, since the geometry of the problems (two dimensions only) and the underlying physical phenomena are much simpler, but we hope that giving rigorous results and methods will prove useful. The geometry under consideration is that of a self-similar ramified bidimensional domain, see Figure 1 below. It can be seen as a simple model for a tree or for lungs. This domain can be obtained by glueing together dilated/translated copies of a simple polygonal domain of $\mathbb{R}^2$, called ω^0 below.

Partial differential equations in domain with fractal boundaries or fractal interfaces is a relatively new topic: variational techniques have been developed, involving new results on fonctional analysis, see [9, 7, 8]. A very nice theory on variational problems in fractal media is given in [10].

The difficulty of solving boundary value problems with partial differential equations in this kind of domains comes essentially from the multiscale character of the boundary. Yet, when the equation is homogeneous, it is possible to make use of the geometric self-similarity in order to compute very accurately the restrictions of the solutions to subdomains obtained by interrupting

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the fractal construction after a finite number of generations.
In a previous work [1], we have considered a Poisson problem with homogeneous Neumann conditions on the whole boundary, except on the bottom part of the boundary (see Figure 1), noted Γ^0 below, where a Dirichlet condition was imposed. For that, it is possible to solve an equivalent boundary value problem in a subdomain obtained by interrupting the fractal construction after a finite number of generations: this equivalent problem involves a nonlocal Dirichlet to Neumann operator T^0, which maps a function defined on Γ^0 to the normal derivative of its harmonic lifting in the whole domain. It turns out that the Dirichlet to Neumann operator on Γ^0 can be computed very accurately by making use of the geometric self-similarity. The Dirichlet to Neumann operator is approximated as the limit of an inductive sequence, see (37) (38) (39) below. When discretizing the problem with finite elements with self similar meshes, the same procedure can be implemented. The numerical method developed in [1] is reminiscent of some of the techniques involved in the theoretical analysis of finitely ramified fractals (see [12],[15], [14], [13], and [2, 11, 6] for numerical simulations). The simple structure of these sets allows to do an explicit analysis of the spectral properties. This involves the dynamics of a renormalization map which acts on the Dirichlet to Neumann operator on the boundary (which for finitely ramified fractal consists only on a finite number of points). Here, the natural boundary is not so simple, but the numerical method is based on a similar strategy.
In the present paper, we are interested in solving the same kind of problem, except that the Neumann data is nonzero on the top part of the boundary, called Γ^∞ below. The first thing to do is to give a meaning to this kind of problem. For that, it is necessary to prove nonstandard extension and trace theorems, which are, in our opinion, interesting by themselves. This is done in § 3.
Next, an interesting problem is to design a method which permits to approximate numerically the restriction of the solution to a subdomain obtained by interrupting the fractal construction after a finite number of generations. This will be done by expanding the Neumann data on Γ^∞ on the basis of Haar wavelets. hen the Neumann data is a Haar wavelet, the solution of the boundary problem can be computed by using the operator T^0 mentioned above, thanks to self-similarity. The program described above is carried out at a continuous level in §4, and at a discrete level in § 5, where finite element are used with special self-similar meshes. Finally, numerical examples are given in § 6, with reults in very good agrement with the theory.

2 Geometrical setting of the model problem

Consider the following T-shaped subset of $\mathbb{R}^2$

$$Q^0 = \big((-1,1)\times(0,2]\big) \cup ((-2,2)\times(2,3)) \cup (((-2,-1)\cup(1,2))\times\{3\}) .$$

The fractal domain Ω^0 is constructed as an infinite union of subsets of $\mathbb{R}^2$ obtained by translating/dilating Q^0; at a first stage, two copies of $1/2\cdot Q^0$ are translated respectively on top-left and on top-right of Q^0 and are glued to Q^0: more precisely, let F_1 and F_2 be the affine mappings

$$F_i(x) = \xi_i^1 + \frac{1}{2}x, \text{ where } \xi_1^1 = (-\frac{3}{2}, 3) \text{ and } \xi_2^1 = (\frac{3}{2}, 3), \tag{1}$$

and let Q^1 be the set $Q^1 = F_1(Q^0) \cup F_2(Q^0)$. Next, the construction is recursive: the points ξ_i^n for $i = 1, \dots, 2^n$ are defined by the relation: for $j = 1, \dots, 2^{n-1}$, $\xi_{2j-1}^n = \xi_j^{n-1} + \frac{1}{2^{n-1}}\xi_1^1$ and $\xi_{2j}^n = \xi_j^{n-1} + \frac{1}{2^{n-1}}\xi_2^1$, and the following sets are introduced:

$$Q^n = \cup_{i=1}^{2^n} Q_i^n, \quad \text{with} \quad Q_i^n = \xi_i^n + \frac{1}{2^n}\cdot Q^0. \tag{2}$$

For $n \geq 1$, calling $\mathcal{A}_n$ the set containing all the mappings from $\{1, \dots, 2^{n-1}\}$ to $\{1, 2\}$, and for $\sigma \in \mathcal{A}_n$, $\mathcal{M}_\sigma(F_1, F_2) = F_{\sigma(1)} \circ F_{\sigma(2)} \circ \dots \circ F_{\sigma(2^n)}$, (2) can also be written

$$Q^n = \cup_{\sigma \in \mathcal{A}_n} \mathcal{M}_\sigma(F_1, F_2)(Q^0).$$

It will sometimes be convenient to agree that $\mathcal{A}_0 = \{0\}$ and that $\mathcal{M}_0(F_1, F_2)$ is the identity. Finally, the fractal tree Ω^0 is defined by

$$\Omega^0 = \cup_{n=0}^{\infty} Q^n. \tag{3}$$

The construction of Ω^0 is displayed on Figure 1. It is straightforward to see that $\Omega^0 \subset (-3, 3) \times$

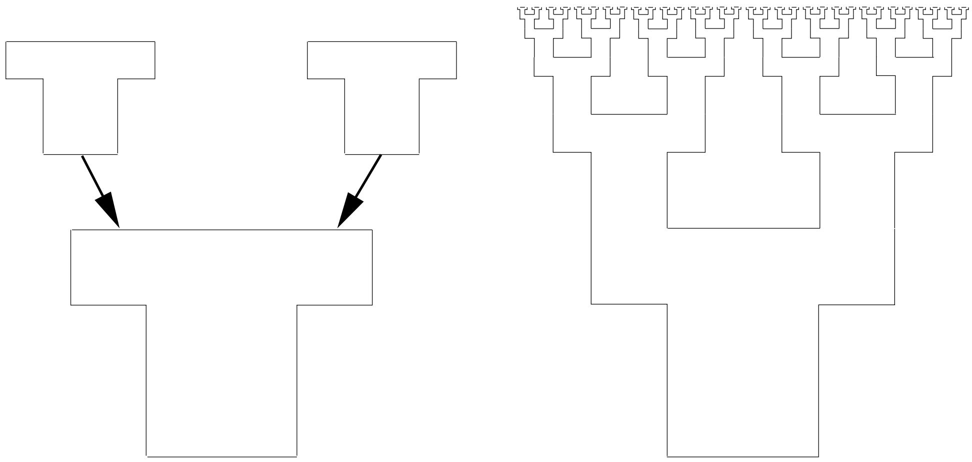

Figure 1: Left: the first step of the construction. Right: the fractal tree (only a few generations are displayed)

$(0, 6)$. Note that Ω^0 may also be obtained as a union of overlapping open subsets of $\mathbb{R}^2$, thus Ω^0 is an open set.
It will be useful to define the truncated fractal Ω^N:

$$\Omega^N = \cup_{n=N}^{\infty} Q^n. \tag{4}$$

The following self-similarity property is true: Ω^N is the union of 2^N translated copies of $\frac{1}{2^N} \cdot \Omega^0$, i.e.

$$\Omega^N = \cup_{\sigma \in \mathcal{A}_N} \Omega^\sigma, \tag{5}$$

where

$$\Omega^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Omega^0). \tag{6}$$

Also, $\Omega^N \backslash \Omega^{N+1} = Q^N$ for any $N \geq 0$.
We define the bottom boundary of Ω^0 by $\Gamma^0 = ((-1; 1) \times \{0\})$ and $\Sigma^0 = \partial\Omega^0 \cap \{(x_1, x_2); x_1 \in \mathbb{R}, 0 < x_2 < 6\}$. We have

$$\partial\Omega^0 \cap \{(x_1, x_2); x \in \mathbb{R}, x_2 < 6\} = \Gamma^0 \cup \Sigma^0. \tag{7}$$

Similarly, the bottom boundary of Ω^N is $\Gamma^N = \cup_{i=1}^{2^N} \Gamma_i^N$, $\Gamma_i^N = \xi_i^N + \frac{1}{2^N} \cdot \Gamma^0$. In an equivalent manner,

$$\Gamma^N = \cup_{\sigma \in \mathcal{A}_N} \Gamma^\sigma, \tag{8}$$

where

$$\Gamma^\sigma = \mathcal{M}_\sigma(F_1, F_2)(\Gamma^0). \tag{9}$$

For $N > 0$, Γ^N is contained in the line $x_2 = 3\sum_{i=0}^{N-1} 2^{-i}$. We define also $\Sigma^N = \partial\Omega^N \cap \{(x_1, x_2); x_1 \in \mathbb{R}, 3\sum_{i=0}^{N-1} 2^{-i} < x_2 < 6\}$. Calling $\Gamma^\infty = [-3:3] \times \{6\}$, one can check easily that

$$\partial\Omega^0 = \Gamma^0 \cup \Sigma^0 \cup \Gamma^\infty. \tag{10}$$

The aim of this paper is to study boundary value problems in Ω^0 with nonhomogeneous Neumann boundary condition on Γ^∞
For what follows, it is also useful to introduce the open domains ω^N, for $N \geq 0$:

$$\omega^N = \mathrm{Int}\left(\Omega^0 \backslash \Omega^{N+1}\right). \tag{11}$$

Remark 1 *Note that it is also possible to construct very similar fractal trees using dilations with ratii α^n with $\alpha \in]0; 1/2]$; here we have chosen $\alpha = 1/2$.*

3 Some functions spaces

Let q be a real number such that $q \geq 1$. Consider the function space $W^{1,q}(\Omega^n) = \{v \in L^q(\Omega^n) \text{ s.t. } \nabla v \in (L^q(\Omega^n))^2\}$. Similarly, for all positive integer p, it is possible to define $W^{p,q}(\Omega^n)$ as the space of functions whose partial derivatives up to order p belong to $L^q(\Omega)$, and for all positive real number $s \notin \mathbb{N}$, $W^{s,q}(\Omega^n)$ is defined by interpolation between $W^{p,q}(\Omega^n)$ and $W^{p+1,q}(\Omega^n)$, where p is the integer such that $p \leq s < p+1$. Likewise, it is possible to define the Sobolev spaces $W^{s,q}(\omega^n)$ for all nonnegative integers n. All the spaces introduced below endowed with their natural norms are Banach spaces. For general results on Sobolev spaces for domains with Lipschitz regular boundaries (which is not the case here), see [3, 4]. In the case $q = 2$, the spaces are Hilbert spaces, and we use the special notation $H^s(\Omega^0) = W^{s,2}(\Omega^0)$.
Of course, for all $n \geq 0$, the restriction of a function $v \in H^1(\Omega^0)$ to ω^n belongs to $H^1(\omega^n)$, so it is possible to define the trace of v on Γ^n. The trace operator on Γ^n is bounded from $H^1(\Omega^0)$ to $L^2(\Gamma^n)$, so one can define the closed subspace of $H^1(\Omega^n)$:

$$\mathcal{V}(\Omega^n) = \{v \in H^1(\Omega^n) \text{ s.t. } v|_{\Gamma^n} = 0\}. \tag{12}$$

In what follows, for a function u integrable on Γ^σ, the notation $\langle u \rangle_{\Gamma^\sigma}$ will be used for the mean value of u on Γ^σ.
We will also use the notation $\lesssim$ to indicate that there may arise constants in the estimates, which are independent of the index n in Ω^n or ω^n or on the mesh size when dealing with finite elements.

3.1 Poincaré's inequality and consequences

By generalizing slightly the results proved in [1], we have the

Theorem 1 *For $p \in \mathbb{R}$, $1 \leq p$ and any function $u \in W^{1,p}(\Omega^0)$ whose trace on Γ^0 is zero,*

$$\|u\|_{L^p(\Omega^0)} \leq 8p^{-\frac{1}{p}} \|\nabla u\|_{L^p(\Omega^0)}. \tag{13}$$

There exists a positive constant C such that

- *for all $n \geq 0$ and for all $u \in W^{1,p}(\Omega^0)$,*

$$\|u\|^p_{L^p(\Omega^n)} \leq C\left(2^{-np}\|\nabla u\|^p_{L^p(\Omega^n)} + 2^{-n}\|u|_{\Gamma^n}\|^p_{L^p(\Gamma^n)}\right), \tag{14}$$

- *for all integers n, q, $n > q \geq 0$ and for all $v \in W^{1,p}(\Omega^0)$,*

$$\left|\|v|_{\Gamma^q}\|^p_{L^p(\Gamma^q)} - \|v|_{\Gamma^n}\|^p_{L^p(\Gamma^n)}\right| \leq C2^{(1-p)q}\|\nabla v\|^p_{L^p(\omega^n\backslash\omega^q)}. \tag{15}$$

- *for all $u \in W^{1,p}(\Omega^0)$, for all $N \geq 0$,*

$$\|u\|^p_{L^p(\Omega^N)} \leq C2^{-N}\left(\|\nabla u\|^p_{L^p(\Omega^0)} + \|u|_{\Gamma^0}\|^p_{L^p(\Gamma^0)}\right).$$

The imbedding from $W^{1,p}(\Omega^0)$ in $L^p(\Omega^0)$ is compact.

3.2 Extensions and traces

3.2.1 Orientation

We aim at constructing an extension operator mapping a function of $W^{1,q}(\Omega^0)$, $q \geq 1$, to a function defined in a simple polygonal domain of $\mathbb{R}^2$. Call $\tilde{Q}^0$ of $\mathbb{R}^2$ the convex hull of the points

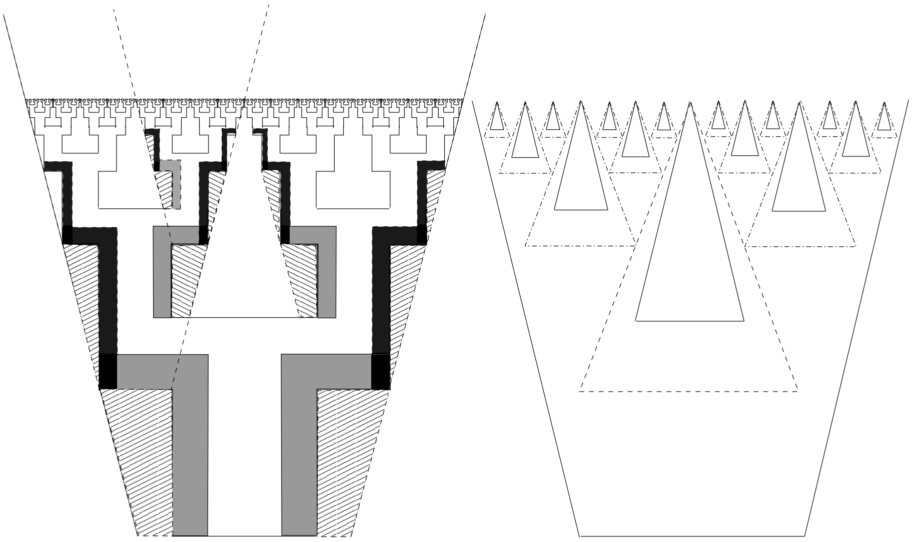

Figure 2: The extension is performed in two steps

$(-\frac{3}{2}, 0)$, $(\frac{3}{2}, 0)$, $(-\frac{5}{2}, 3)$, $(\frac{5}{2}, 3)$, and $\tilde{\Omega}^0$ the new fractal domain

$$\tilde{\Omega}^0 = \bigcup_{n\in\mathbb{N}} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(F_1, F_2)\tilde{Q}^0.$$

It is represented on the right part of Figure 2 with full lines. It is clear that $\Omega^0 \subset \tilde{\Omega}^0$.
We call also $\widehat{\Omega}^0$ the convex hull of the points $(-\frac{3}{2}, 0), (\frac{3}{2}, 0), (-3, 6), (3, 6)$. Note that $\tilde{\Omega}^0$ is obtained by removing an infinite family of nonoverlapping triangles from $\widehat{\Omega}^0$. More precisely,

consider the triangle $T = \operatorname{conv}((0,6),(1,3),(-1,3)) \subset \overline{\widehat{\Omega}^0}\backslash\tilde{\Omega}^0$, which is visible on the right part of Figure 2 and call G_1, G_2 the affine mappings $G_1(x) = (1+\frac{x_1}{2}, 3+\frac{x_2}{2})$, $G_2(x) = (-1+\frac{x_1}{2}, 3+\frac{x_2}{2})$. We have that

$$\tilde{\Omega}^0 = \widehat{\Omega}^0 \backslash \left(\bigcup_{n\in\mathbb{N}} \bigcup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(G_1, G_2)T \right).$$

3.2.2 Bounded extension from $W^{1,q}(\Omega^0)$ to $W^{1,q}(\tilde{\Omega}^0)$, $q \le 2$

Our first goal is to construct an extension operator, bounded from $W^{1,q}(\Omega^0)$ to $W^{1,q}(\tilde{\Omega}^0)$, $q, 1 \le q \le 2$,. This is done in two steps.

First step Call D_L the L-shaped compact set $D_L = \operatorname{conv}\left((-1,-2),(-\frac{1}{2},-2),(-\frac{1}{2},2),(-1,2)\right) \cup \operatorname{conv}\left((-\frac{1}{2},2),(-2,2),(-2,\frac{5}{2}),(-\frac{1}{2},\frac{5}{2})\right)$ and D_R the image of D_L by the symmetry of axis $x_1 = 0$. Call also T_L the triangle $\operatorname{conv}((-1,-2),(-1,2),(-2,2))$ and T_R the image of T_L by the symmetry of axis $x_1 = 0$.
It is possible to construct an extension operator $\mathcal{E}_L$, which maps continuously $W^{1,q}(D_L)$ to $W^{1,q}(D_L \cup T_L)$, for all $q \in [1,2]$: for all $q \in [1,2]$, there is a positive constant c such that, for all $u \in W^{1,q}(D_L)$, $\|\mathcal{E}_L u\|_{W^{1,q}(T_L)} \le c\|u\|_{W^{1,q}(D_L)}$.
Call $\Omega^0_L = \Omega^0 \cap \bigcup_{n\in\mathbb{N}} F_1^n(D_L)$ and $\tilde{\Omega}^0_L = \tilde{\Omega}^0 \cap \bigcup_{n\in\mathbb{N}} F_1^n(D_L \cup T_L)$. Similarly $\Omega^0_R = \Omega^0 \cap \bigcup_{n\in\mathbb{N}} F_2^n(D_R)$ and $\tilde{\Omega}^0_R = \tilde{\Omega}^0 \cap \bigcup_{n\in\mathbb{N}} F_2^n(D_R \cup T_R)$. The previous observation and the facts that

- $|F_1^n(T_L) \cap F_1^m(T_L)| = 0$ if $n \ne m$.
- for any point x in Ω^0_L, there exist at least one and at most two integers n such that $x \in F_1^n(D_L)$.

enable to construct an extension operator $\tilde{\mathcal{E}}_L$, bounded from $W^{1,q}(\Omega^0_L)$ to $W^{1,q}(\tilde{\Omega}^0_L)$, $1 \le q \le 2$, by:

$$\begin{aligned} &\text{if } x \in F_1^n(T_L),\ n \ge 1, \text{ then } (\tilde{\mathcal{E}}_L u)(x) = \mathcal{E}_L((u \circ F_1^n)|_{D_L})\left((F_1^n)^{-1}(x)\right), \\ &\text{if } x \in \tilde{\Omega}^0 \cap T_L, \text{ then } (\tilde{\mathcal{E}}_L u)(x) = \mathcal{E}_{L,0}(u|_{D_L})(x), \end{aligned}$$

where $\mathcal{E}_{L,0}$ is any extension operator which maps continuously $W^{1,q}(\Omega^0 \cap D_L)$ to $W^{1,q}(\tilde{\Omega}^0 \cap (D_L \cup T_L))$, $1 \le q \le 2$.
By symmetry, it possible to construct an extension operator, bounded from $\tilde{\mathcal{E}}_R$ from $W^{1,q}(\Omega^0_R)$ to $W^{1,q}(\tilde{\Omega}^0_R)$, $1 \le q \le 2$.

Second step Let $\mathcal{G}_L = \{F_2 \circ \mathcal{M}_\sigma(F_1, F_2), \sigma \in \mathcal{A}_n, n \in \mathbb{N}\}$ and $\mathcal{G}_R = \{F_1 \circ \mathcal{M}_\sigma(F_1, F_2), \sigma \in \mathcal{A}_n, n \in \mathbb{N}\}$. We observe that for any point x in $\tilde{\Omega}^0\backslash\Omega^0$, one and only one of the four successive items is true:

1. either $x \in \tilde{\Omega}^0_L\backslash\Omega^0$,
2. or $x \in \tilde{\Omega}^0_R\backslash\Omega^0$,
3. or there exists a unique tranformation τ in $\mathcal{G}_L$ such that $x \in \tau(\tilde{\Omega}^0_L\backslash\Omega^0)$,
4. or there exists a unique tranformation τ in $\mathcal{G}_R$ such that $x \in \tau(\tilde{\Omega}^0_R\backslash\Omega^0)$,

From this observation, it is possible to construct an extension operator $\mathcal{E}$, bounded from $W^{1,q}(\Omega^0)$ to $W^{1,q}(\tilde{\Omega}^0)$, $1 \le q \le 2$, by: for $x \in \tilde{\Omega}^0\backslash\Omega^0$,

- if item 1 is true: $(\mathcal{E}u)(x) = (\tilde{\mathcal{E}}_L u)(x)$,
- if item 2 is true: $(\mathcal{E}u)(x) = (\tilde{\mathcal{E}}_R u)(x)$,
- if item 3 is true: $(\mathcal{E}u)(x) = (\tilde{\mathcal{E}}_L(u \circ \tau))(\tau^{-1}(x))$,
- if item 4 is true: $(\mathcal{E}u)(x) = (\tilde{\mathcal{E}}_R(u \circ \tau))(\tau^{-1}(x))$,

3.2.3 Bounded extension from $W^{1,q}(\tilde{\Omega}^0)$ to $W^{1,q}(\widehat{\Omega}^0)$, $1 \le q < 2$

Consider the triangle $\widehat{T} = \mathrm{conv}((0,6), (\frac{3}{2}, 2), (-\frac{3}{2}, 2))$ which is displayed on the left of Figure 2, with interrupted lines. The key observation is that

$$\widehat{T} \backslash T \subset \tilde{\Omega}^0, \tag{16}$$

and that

$$\forall \sigma_1, \sigma_2 \in \bigcup_{n \in \mathbb{N}} \mathcal{A}_n \text{ such that } \sigma_1 \neq \sigma_2, \quad \mathcal{M}_{\sigma_1}(G_1, G_2)(\widehat{T}) \cap \mathcal{M}_{\sigma_2}(G_1, G_2)(\widehat{T}) = \emptyset. \tag{17}$$

It is well known that there exists a extension operator $\tilde{\mathcal{F}}$, bounded from $W^{1,q}(\widehat{T} \backslash T)$ to $W^{1,q}(\widehat{T})$, for any $q, 1 \le q < 2$. Note that the operator cannot be bounded in from $W^{1,2}(\widehat{T} \backslash T)$ to $W^{1,2}(\widehat{T})$, because its construction involves polar coordinates around the common vertex of T and $\widehat{T}$. From this and (17), we can construct an extension operator $\mathcal{F}$, bounded from $W^{1,q}(\tilde{\Omega}^0)$ to $W^{1,q}(\widehat{\Omega}^0)$, $1 \le q < 2$ by

$$\text{if } x \in \mathcal{M}_\sigma(G_1, G_2)(T), \quad \mathcal{F}(u)(x) = \tilde{\mathcal{F}}\left((u \circ \mathcal{M}_\sigma(G_1, G_2)) |_{\widehat{T} \backslash T} \right) \circ (\mathcal{M}_\sigma(G_1, G_2))^{-1}(x).$$

3.2.4 Bounded extension from $W^{1,q}(\Omega^0)$ to $W^{1,q}(\widehat{\Omega}^0)$, $1 \le q < 2$

By composing the extension operators $\mathcal{E}$ and $\mathcal{F}$ constructed above, we have proved the

Theorem 2 *There exists an extension operator $\mathcal{J}$ bounded from $W^{1,q}(\Omega^0)$ to $W^{1,q}(\hat{\Omega}^0)$, for all q, $1 \le q < 2$,*

Remark 2 *Of course, since Ω^0 is a bounded domain, we have also that for q, $1 \le q < 2$, the extension operator $\mathcal{J}$ is bounded from $H^1(\Omega^0)$ to $W^{1,q}(\hat{\Omega}^0)$.*

As a consequence of Theorem 2, we have the Sobolev imbeddings:

Proposition 1 (Sobolev imbeddings) *Let q be a real number such that $1 \le q < 2$: we have the continuous imbeddings:*

$$W^{1,q}(\Omega^0) \subset L^p(\Omega^0), \quad \forall p, 1 \le p \le q^*, \quad q^* = \frac{2q}{2-q},$$

and the imbedding is compact if $p < q^*$.
Furthermore, for all q, p, $1 \le q < 2$, $1 \le p \le q^*$, there exists a constant C such that for all $N \ge 0$,

$$\|u\|^p_{L^p(\Omega^N)} \le C \left(2^{\frac{2N(p-q)-qpN}{q}} \|\nabla u\|^p_{L^q(\Omega^N)} + 2^{\frac{N(p-2q)}{q}} \|u\|^p_{L^q(\Gamma^N)} \right). \tag{18}$$

For all real number $p \ge 1$, $H^1(\Omega^0) \subset L^p(\Omega^0)$ with continuous and compact imbedding.

3.2.5 Density results

Theorem 3 *For $q, 1 \leq q < 2$, the space $\mathcal{C}^\infty(\overline{\Omega^0})$ is dense in $W^{1,q}(\Omega^0)$, and there exists a sequence of linear operators $(\mathcal{S}_n)_{n\in\mathbb{N}}$ from $W^{1,q}(\Omega^0)$ to $\mathcal{C}^\infty(\overline{\widehat{\Omega}^0})$ such that*

$$\forall u \in W^{1,q}(\Omega^0), \quad \lim_{n\to\infty} \|u - (\mathcal{S}_n u)|_{\Omega^0}\|_{W^{1,q}(\Omega^0)} = 0, \tag{19}$$

and for a constant c,

$$\forall u \in W^{1,q}(\Omega^0), \quad \|\mathcal{S}_n u\|_{W^{1,q}(\widehat{\Omega}^0)} \leq c\|u\|_{W^{1,q}(\Omega^0)}. \tag{20}$$

Proof. We know that $\mathcal{C}^\infty(\overline{\widehat{\Omega}^0})$ is dense in $W^{1,q}(\widehat{\Omega}^0)$ and that there exists a sequence of linear operators $(\widehat{\mathcal{S}}_n)_{n\in\mathbb{N}}$ from $W^{1,q}(\widehat{\Omega}^0)$ to $\mathcal{C}^\infty(\overline{\widehat{\Omega}^0})$ such that

$$\forall u \in W^{1,q}(\widehat{\Omega}^0), \quad \lim_{n\to\infty} \|u - \widehat{\mathcal{S}}_n u\|_{W^{1,q}(\widehat{\Omega}^0)} = 0,$$

and for a constant c,

$$\forall u \in W^{1,q}(\widehat{\Omega}^0), \quad \|\widehat{\mathcal{S}}_n u\|_{W^{1,q}(\widehat{\Omega}^0)} \leq c\|u\|_{W^{1,q}(\widehat{\Omega}^0)}.$$

Consider an extension operator $\mathcal{J}$ as in Theorem 2. The operators $\mathcal{S}_n = \widehat{\mathcal{S}}_n \circ \mathcal{J}$ answer the question. ∎

3.2.6 Traces

Take q, $1 < q \leq 2$ and call $\mathcal{N}_p^q$ the mapping

$$\mathcal{N}_p^q : W^{1,q}(\Omega^0) \to \mathbb{R}_+, \quad \mathcal{N}_p^q(v) = \|v|_{\Gamma^p}\|^q_{L^q(\Gamma^p)}. \tag{21}$$

Estimate (15) tells us that for any $v \in W^{1,q}(\Omega^0)$, $(\mathcal{N}_p^q(v))_{p\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}_+$ and that it converges to a real number $\mathcal{N}_\infty^q(v)$. It is an easy matter to prove the following

Lemma 1 *Let q be a real number such that $1 < q \leq 2$. The mapping $v \mapsto \mathcal{N}_\infty^q(v)$ is homogeneous of degree q.*
There exists a constant C such that forall $v \in W^{1,q}(\Omega^0)$,

$$\mathcal{N}_\infty^q(v) \leq C\|v\|^q_{W^{1,q}(\Omega^0)}. \tag{22}$$

For $v \in \mathcal{C}^\infty(\overline{\widehat{\Omega}^0})$,

$$\mathcal{N}_\infty^q(v|_{\Omega^0}) = \frac{1}{3}\|v\|^q_{L^q(\Gamma^\infty)}. \tag{23}$$

Theorem 4 *Let q be a real number such that $1 < q < 2$. There is a continuous trace operator γ from $W^{1,q}(\Omega^0)$ onto $W^{1-\frac{1}{q},q}(\Gamma^\infty)$, such that for all $u \in \mathcal{C}^\infty(\overline{\widehat{\Omega}^0})$,*

$$\gamma(u|_{\Omega^0}) = u|_{\Gamma^\infty}.$$

Proof. Consider a sequence of operators $(\mathcal{S}_n)_{n\in\mathbb{N}}$ as in Theorem 3. Let u be a function in $W^{1,q}(\Omega^0)$. The function $\mathcal{S}_n(u)$ has a trace in $W^{1-\frac{1}{q},q}(\Gamma^\infty)$, and we have from (20) that for a constant c (independent of n)

$$\|\mathcal{S}_n(u)\|_{W^{1-\frac{1}{q},q}(\Gamma^\infty)} \leq c\|u\|_{W^{1,q}(\Omega^0)}.$$

Therefore, one can extract a subsequence $\mathcal{S}_{\phi(n)}$ such that $\mathcal{S}_{\phi(n)}(u)|_{\Gamma^\infty}$ converges weakly in $W^{1-\frac{1}{q},q}(\Gamma^\infty)$ and strongly in $L^q(\Gamma^\infty)$ to some function $w \in W^{1-\frac{1}{q},q}(\Gamma^\infty)$. There remains to prove that w is unique (i.e. the whole sequence converges), and that w depends only on u and not on the sequence $\mathcal{S}_n$.

Assume that there exist two subsequences $(\mathcal{S}_{\phi(n)}(u))_{n\in\mathbb{N}}$ and $(\mathcal{S}_{\psi(n)}(u))_{n\in\mathbb{N}}$ whose traces on Γ^∞ converges respectively to w and w' weakly in $W^{1-\frac{1}{q},q}(\Gamma^\infty)$ and strongly in $L^q(\Gamma^\infty)$. We have from (23) that

$$\begin{aligned}\|w-w'\|^q_{L^q(\Gamma^\infty)} &= \lim_{n\to\infty}\|\mathcal{S}_{\phi(n)}(u)-\mathcal{S}_{\psi(n)}(u)\|^q_{L^q(\Gamma^\infty)}\\ &= 3\lim_{n\to\infty}\mathcal{N}^q_\infty(\mathcal{S}_{\phi(n)}(u)|_{\Omega^0}-\mathcal{S}_{\psi(n)}(u)|_{\Omega^0})\end{aligned}$$

From (22),

$$\|w-w'\|^q_{L^q(\Gamma^\infty)} \le 3C\limsup_{n\to\infty}\|\mathcal{S}_{\phi(n)}(u)|_{\Omega^0}-\mathcal{S}_{\psi(n)}(u)|_{\Omega^0}\|^q_{W^{1,q}(\Omega^0)},$$

and we conclude from (19) that

$$\|w-w'\|^q_{L^q(\Gamma^\infty)} = 0,$$

and therefore $w=w'$.
The same argument leads to the fact that w does not depend on the sequence of operators $(\mathcal{S}_n)_{n\in\mathbb{N}}$.
Therefore the mapping $\gamma : u \mapsto w$ is a continuous linear operator from $W^{1,q}(\Omega^0)$ to $W^{1-\frac{1}{q},q}(\Gamma^\infty)$.
It can also be checked by reproducing the argument above that if $u \in \mathcal{C}^\infty(\overline{\widehat{\Omega}^0})$ then $\gamma(u|_{\Omega^0}) = u|_{\Gamma^\infty}$. Therefore γ is a trace operator from $W^{1,q}(\Omega^0)$ to $W^{1-\frac{1}{q},q}(\Gamma^\infty)$.
It is surjective because the trace operator from $W^{1,q}(\widehat{\Omega}^0)$ to $W^{1-\frac{1}{q},q}(\Gamma^\infty)$ is surjective. ■
From the continuous imbedding $H^1(\Omega^0)\subset W^{1,q}(\Omega^0)$, and from the Sobolev imbedding theorems in dimension one, we have the

Corollary 1 *The trace operator is continuous from $H^1(\Omega^0)$ to $W^{1-\frac{1}{q},q}(\Gamma^\infty)$, for all real numbers q with $1<q<2$. The trace operator is continuous from $H^1(\Omega^0)$ to $L^p(\Gamma^\infty)$, for all real number p with $1\le p<\infty$.*

Corollary 2 *For all $u\in H^1(\Omega^0)$, for all real number $p, p>1$,*

$$\mathcal{N}^p_\infty(u) = \frac{1}{3}\|\gamma(v)\|^p_{L^p(\Gamma^\infty)}. \tag{24}$$

For all $u, v\in H^1(\Omega^0)$,

$$\lim_{n\to\infty}\int_{\Gamma^n} u|_{\Gamma^n} v|_{\Gamma^n} = \frac{1}{3}\int_{\Gamma^\infty}\gamma(u)\gamma(v). \tag{25}$$

4 Poisson problems with nonzero Neumann data on Γ^∞

4.1 The boundary value problem

Let p be a real number greater than one, take $g\in L^p(\Gamma^\infty)$. We are interested in the variational problem

$$\text{find } u\in\mathcal{V}(\Omega^0) \text{ such that and for all } v\in\mathcal{V}(\Omega^0),\quad \int_{\Omega^0}\nabla u\cdot\nabla v = \frac{1}{3}\int_{\Gamma^\infty} g\gamma(v). \tag{26}$$

From Corollary 1, the linear form $v \mapsto \int_{\Gamma^\infty} g\gamma(v)$ is bounded on $H^1(\Omega^0)$, and from Theorem 1, problem (26) has a unique solution.
The next result says that the solution to (26) can be approximated by solving boundary value problems in ω^n, $n \to \infty$:

Proposition 2 *Let p be a real number such that $1 < p \leq 2$. If g is the trace on Γ^∞ of a function $\tilde{g}$ belonging to $W^{1,p}(\widehat{\Omega}^0)$, then*

$$\lim_{n\to\infty} \|u|_{\omega^n} - u_n\|_{H^1(\omega^n)} = 0,$$

where u is defined by (26) and $u_n \in \mathcal{V}(\omega^n)$ is the solution to:

$$\text{for all } v \in \mathcal{V}(\omega^n), \quad \int_{\omega^n} \nabla u_n \cdot \nabla v = \int_{\Gamma^{n+1}} \tilde{g}|_{\Gamma^{n+1}} v|_{\Gamma^{n+1}}. \tag{27}$$

Proof. Calling e_n the error $e_n = u|_{\omega^n} - u_n \in \mathcal{V}(\omega^n)$, we see that

$$\int_{\omega^n} \nabla e_n \cdot \nabla v = \left(\frac{1}{3} \int_{\Gamma^\infty} g\gamma(v) - \int_{\Gamma^{n+1}} \tilde{g}|_{\Gamma^{n+1}} v|_{\Gamma^{n+1}} \right) - \int_{\Omega^{n+1}} \nabla u \cdot \nabla v. \tag{28}$$

It can be proved that for all $v \in \mathcal{V}(\Omega^0)$, $\frac{1}{3} \int_{\Gamma^\infty} g\gamma(v) = \lim_{n\to\infty} \int_{\Gamma^{n+1}} \tilde{g}|_{\Gamma^{n+1}} v|_{\Gamma^{n+1}}$, so the first term in the right hand side tends to zero. More precisely, Proposition 1 tells us that for all r, $1 < r < p$, the function $\tilde{g}v$ belongs to $W^{1,r}(\Omega^0)$. It is easy to check by a scaling argument that, for all $m < m'$,

$$\left| \int_{\Gamma^{m+1}} \tilde{g}|_{\Gamma^{m+1}} v|_{\Gamma^{m+1}} - \int_{\Gamma^{m'+1}} \tilde{g}|_{\Gamma^{m'+1}} v|_{\Gamma^{m'+1}} \right| \lesssim 2^{m\frac{1-r}{r}} \|\nabla(\tilde{g}v)\|_{L^r(\omega^{m'} \setminus \omega^m)}.$$

Since $\tilde{g}$ is fixed, this implies from Proposition 1 that

$$\left| \int_{\Gamma^{m+1}} \tilde{g}|_{\Gamma^{m+1}} v|_{\Gamma^{m+1}} - \int_{\Gamma^{m'+1}} \tilde{g}|_{\Gamma^{m'+1}} v|_{\Gamma^{m'+1}} \right| \lesssim 2^{m\frac{1-r}{r}} \|v\|_{H^1(\Omega^0)}.$$

Thus, the sequence of continuous linear forms on $\mathcal{V}(\Omega^0)$: $v \mapsto \int_{\Gamma^{n+1}} \tilde{g}|_{\Gamma^{n+1}} v|_{\Gamma^{n+1}}$ is a Cauchy sequence in the dual of $\mathcal{V}(\Omega^0)$, and the limit can be identified by using Theorem 3: therefore,

$$\lim_{n\to\infty} \sup_{v \in \mathcal{V}(\Omega^0), v \neq 0} \frac{\left| \frac{1}{3} \int_{\Gamma^\infty} g\gamma(v) - \int_{\Gamma^{n+1}} \tilde{g}|_{\Gamma^{n+1}} v|_{\Gamma^{n+1}} \right|}{\|v\|_{H^1(\Omega^0)}} = 0. \tag{29}$$

We also have that

$$\lim_{n\to\infty} \sup_{v \in \mathcal{V}(\Omega^0), v \neq 0} \frac{\int_{\Omega^{n+1}} \nabla u \cdot \nabla v}{\|v\|_{H^1(\Omega^0)}} = 0. \tag{30}$$

From (28) (29) and (30), we deduce the desired result. ∎
We will try to solve (26) numerically. Of course, it is not possible to represent completely the domain Ω^0 in numerical simulations, because this would imply an infinite memory and computing time. Rather, for some $n \in \mathbb{N}$, we aim at computing as well as possible the restriction of u in (26) to ω^n. This turns out to be possible, but for that, we need to use nonlocal operators on Γ^σ, $\sigma \in \mathcal{A}_{n+1}$. These operators have been studied theoretically and numerically in [1].

4.2 The Dirichlet to Neumann operator

Here we give results which where proved in [1]. For an integer $n \geq 0$, and for $\sigma \in \mathcal{A}_n$, one can define the harmonic lifting operator $\mathcal{H}^\sigma$ from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to $H^1(\Omega^\sigma)$: for all $u \in H^{\frac{1}{2}}(\Gamma^\sigma)$, the trace of $\mathcal{H}^\sigma(u)$ on Γ^σ is u and for all $v \in \mathcal{V}(\Omega^\sigma)$, $\int_{\Omega^\sigma} \nabla \mathcal{H}^\sigma(u) \cdot \nabla v = 0$. Since $\mathcal{A}_0 = \{0\}$, we denote by $\mathcal{H}^0$ the harmonic lifting in Ω^0. It is easy to check that, for all $v \in H^{\frac{1}{2}}(\Gamma^\sigma)$,

$$\mathcal{H}^\sigma(v) \circ \mathcal{M}_\sigma(F_1, F_2) = \mathcal{H}^0(v \circ \mathcal{M}_\sigma(F_1, F_2)). \tag{31}$$

Theorem 5 *There exists a positive constant C such that, for all $u \in H^{\frac{1}{2}}(\Gamma^0)$,*

$$\|\nabla \mathcal{H}^0(u)\|_{L^2(\omega^0)} \geq C \|\nabla \mathcal{H}^0(u)\|_{L^2(\Omega^0)}. \tag{32}$$

There exists a real number ρ, $0 < \rho < 1$ such that for all $u \in H^{\frac{1}{2}}(\Gamma^0)$,

$$\int_{\Omega^N} |\nabla \mathcal{H}^0(u)|^2 \leq \rho^N \int_{\Omega^0} |\nabla \mathcal{H}^0(u)|^2. \tag{33}$$

For $\sigma \in \mathcal{A}_n$, one can define the operators T^σ, from $H^{\frac{1}{2}}(\Gamma^\sigma)$ to their respective duals by $\langle T^\sigma u, v\rangle = \int_{\Omega^\sigma} \nabla \mathcal{H}^\sigma(u) \cdot \nabla \mathcal{H}^\sigma(v) = \int_{\Omega^\sigma} \nabla \mathcal{H}^\sigma(u) \cdot \nabla \tilde{v}$, for any function $\tilde{v} \in H^1(\Omega^\sigma)$ such that $\tilde{v}|_{\Gamma^\sigma} = v$. From the self-similarity of Ω^0, we have that

$$\forall u, v \in H^{\frac{1}{2}}(\Gamma^\sigma), \quad \langle T^\sigma u, v\rangle = \langle T^0 \left(u \circ \mathcal{M}_\sigma(F_1, F_2)\right), \left(v \circ \mathcal{M}_\sigma(F_1, F_2)\right)\rangle, \tag{34}$$

where the duality pairing in left (resp., right) hand side of (34) is the duality $\left(H^{\frac{1}{2}}(\Gamma^\sigma)\right)'$- $H^{\frac{1}{2}}(\Gamma^\sigma)$ (resp., $\left(H^{\frac{1}{2}}(\Gamma^0)\right)'$- $H^{\frac{1}{2}}(\Gamma^0)$).

Lemma 2 *For all $u \in H^{\frac{1}{2}}(\Gamma^0)$, for $n \geq 1$, the restriction of $\mathcal{H}^0(u)$ to ω^{n-1} is the solution to the following boundary value problem: find $\hat{u} \in H^1(\omega^{n-1})$ such that $\hat{u}|_{\Gamma^0} = u$ and $\forall v \in \mathcal{V}(\omega^{n-1})$,*

$$\int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v + \sum_{\sigma \in \mathcal{A}_n} \left\langle T^0 \left(\hat{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\right), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\right\rangle = 0. \tag{35}$$

Furthermore, $\forall v \in H^1(\omega^{n-1})$,

$$\begin{aligned} \langle T^0 u, v|_{\Gamma^0}\rangle &= \int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v + \sum_{\sigma \in \mathcal{A}_n} \langle T^\sigma \hat{u}|_{\Gamma^\sigma}, v|_{\Gamma^\sigma}\rangle \\ &= \int_{\omega^{n-1}} \nabla \hat{u} \cdot \nabla v + \sum_{\sigma \in \mathcal{A}_n} \left\langle T^0(\hat{u}|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\right\rangle. \end{aligned} \tag{36}$$

Lemma 2, in the case $n = 1$, leads us to introduce the cone $\mathbb{O}$ of self adjoint, positive semi-definite, bounded linear operators from $H^{\frac{1}{2}}(\Gamma^0)$ to its dual, vanishing on the constants, and the mapping $\mathbb{M} : \mathbb{O} \mapsto \mathbb{O}$ defined as follows: for $Z \in \mathbb{O}$, define $\mathbb{M}(Z)$ by

$$\forall u \in H^{\frac{1}{2}}(\Gamma^0), \forall v \in H^1(\omega^0), \qquad \langle \mathbb{M}(Z)u, v|_{\Gamma^0}\rangle = \int_{\omega^0} \nabla \hat{u} \cdot \nabla v + \sum_{i=1}^{2} \left\langle Z(\hat{u}|_{\Gamma_i^1} \circ F_i), v|_{\Gamma_i^1} \circ F_i \right\rangle, \tag{37}$$

where $\hat{u} \in H^1(\omega^0)$ is such that $\hat{u}|_{\Gamma^0} = u$ and

$$\forall v \in \mathcal{V}(\omega^0), \qquad \int_{\omega^0} \nabla \hat{u} \cdot \nabla v + \sum_{i=1}^{2} \left\langle Z(\hat{u}|_{\Gamma_i^1} \circ F_i), v|_{\Gamma_i^1} \circ F_i \right\rangle = 0. \tag{38}$$

Lemma 2 tells that T^0 is a fixed point of $\mathbb{M}$. In fact, we have the

Theorem 6 *The operator T^0 is the unique fixed point of $\mathbb{M}$. Moreover, for all $Z \in \mathbb{O}$, there exists a positive constant C independent of n such that, for all $n \geq 0$,*

$$\|\mathbb{M}^n(Z) - T^0\| \leq C\rho^{\frac{n}{4}}, \tag{39}$$

where ρ, $0 < \rho < 1$ is the constant appearing in Theorem 5.

4.3 Solving (26) with $g = 1$

Let $u_F \in \mathcal{V}(\Omega^0)$ be such that for all $v \in \mathcal{V}(\Omega^0)$,

$$\int_{\Omega^0} \nabla u_F \cdot \nabla v = \frac{1}{3}\int_{\Gamma^\infty} v. \tag{40}$$

Let $y \in (H^{\frac{1}{2}}(\Gamma^0))'$ be the normal derivative of u_F on Γ^0, defined by : for all $v \in H^1(\Omega^0)$,

$$\langle y, v\rangle = \int_{\Omega^0} \nabla u_F \cdot \nabla v - \frac{1}{3}\int_{\Gamma^\infty} v. \tag{41}$$

It can be checked that for all $n > 0$, for $\sigma \in \mathcal{A}_n$, $u_F^\sigma = u_F \circ (\mathcal{M}_\sigma(F_1, F_2))^{-1}$ satisfies: $u_F^\sigma \in \mathcal{V}(\Omega^\sigma)$ and for all $v \in \mathcal{V}(\Omega^\sigma)$,

$$\int_{\Omega^\sigma} \nabla u_F^\sigma \cdot \nabla v = \frac{1}{3}\int_{\Gamma^\infty} v \circ \mathcal{M}_\sigma(F_1, F_2). \tag{42}$$

Call now $\tilde{u}_F^\sigma \in \mathcal{V}(\Omega^\sigma)$, the solution to the following problem:

$$\text{for all } v \in \mathcal{V}(\Omega^\sigma), \quad \int_{\Omega^\sigma} \nabla \tilde{u}_F^\sigma \cdot \nabla v = \frac{1}{3}\int_{\Gamma^\infty \cap \overline{\Omega^\sigma}} v. \tag{43}$$

It is easy to prove that

$$\tilde{u}_F^\sigma = \frac{1}{2^n} u_F^\sigma. \tag{44}$$

Let us call $\tilde{u}_F$ the function defined by

$$\begin{aligned} \tilde{u}_F|_{\omega^0} &= 0, \\ \tilde{u}_F|_{\Omega^\sigma} &= \tilde{u}_F^\sigma, \quad \sigma \in \mathcal{A}_1. \end{aligned} \tag{45}$$

From the definition of $\tilde{u}_F^\sigma$, we see that $\tilde{u}_F \in \mathcal{V}(\Omega^0)$, and we can check that for all $v \in \mathcal{V}(\Omega^0)$,

$$\int_{\Omega^0} \nabla \tilde{u}_F \cdot \nabla v = \frac{1}{3}\int_{\Gamma^\infty} v + \frac{1}{2}\sum_{i=1}^{2} \left\langle y, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle. \tag{46}$$

Calling e the error $e = u_F - \tilde{u}_F$, we have: for all $v \in \mathcal{V}(\Omega^0)$,

$$\int_{\Omega^0} \nabla e \cdot \nabla v = -\frac{1}{2}\sum_{i=1}^{2} \left\langle y, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle. \tag{47}$$

Since $e|_{\Omega^1}$ is harmonic, and e coincides with u_F in ω^0, we have that for all $i = 1, 2$ and $v \in H^1(F_i(\Omega^0))$,

$$\int_{F_i(\Omega^0)} \nabla e \cdot \nabla v = \left\langle T^0 \left(u_F \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle. \tag{48}$$

From the fact that e and u_F coincide in ω^0, from (47) and (48), we obtain that $u_F|_{\omega^0}$ is the unique function in $\mathcal{V}(\omega^0)$ such that: for $v \in \mathcal{V}(\omega^0)$,

$$\int_{\omega^0} \nabla u_F \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0 \left(u_F \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle = \int_{\Omega^0} \nabla e \cdot \nabla v = -\frac{1}{2} \sum_{i=1}^{2} \left\langle y, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle, \quad (49)$$

and from (41), y satisfies: for all $v \in H^1(\omega^0)$,

$$\langle y, v \rangle = \int_{\omega^0} \nabla u_F \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0 \left(u_F \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle + \frac{1}{2} \sum_{i=1}^{2} \left\langle y, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle. \quad (50)$$

Note that (50) is trivially (for any y) satisfied if v is constant, because T^0 is symmetric and $T^0 1 = 0$. On the other hand, from (41), y satisfies also:

$$\langle y, 1 \rangle = -\frac{1}{3} |\Gamma^\infty| = -2. \quad (51)$$

Conversely, for $z \in (H^{\frac{1}{2}}(\Gamma^0))'$, call $U(z) \in \mathcal{V}(\omega^0)$ the function uniquely defined by: for $v \in \mathcal{V}(\omega^0)$,

$$0 = \int_{\omega^0} \nabla U(z) \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0 \left(U(z) \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle + \frac{1}{2} \sum_{i=1}^{2} \left\langle z, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle. \quad (52)$$

Regularity results for Laplace's equation imply the following Lemma

Lemma 3 *For all $z \in (H^{\frac{1}{2}}(\Gamma^0))'$, $U(z) \in H^{\frac{3}{2}}(\omega^0 \cap \{x_2 < 1\})$.*

Notice that, thanks to (52), for $v \in H^1(\omega^0)$,

$$\int_{\omega^0} \nabla U(z) \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0 \left(\left(U(z) \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle + \frac{1}{2} \sum_{i=1}^{2} \left\langle z, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle$$

depends on $v|_{\Gamma^0}$ only. From Lemma 3, we see that this quantity is in fact $\int_{\Gamma^0} \frac{\partial U(z)}{\partial n}|_{\Gamma^0} v|_{\Gamma^0}$. Therefore, one can define the operator B^0 bounded from $(H^{\frac{1}{2}}(\Gamma^0))'$ to $L^2(\Gamma^0)$ by

$$B^0 z = \frac{\partial U(z)}{\partial n}|_{\Gamma^0},$$

or in an equivalent manner, $\forall v \in H^1(\omega^0)$,

$$\langle B^0 z, v|_{\Gamma^0} \rangle = \int_{\omega^0} \nabla U(z) \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0 \left(\left(U(z) \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle + \frac{1}{2} \sum_{i=1}^{2} \left\langle z, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle, \quad (53)$$

and we have

$$\forall z \in H^{\frac{1}{2}}(\Gamma^0), \qquad \langle B^0 z, 1 \rangle = \langle z, 1 \rangle.$$

From (49), (50), (51), and Lemma 3, we have

Lemma 4 *The distribution y defined in (41) satisfies the equations:*

$$\begin{aligned} \langle y - B^0 y, v \rangle =& 0, \quad \forall v \in H^{\frac{1}{2}}(\Gamma^0), \\ \langle y, 1 \rangle =& -2, \end{aligned} \quad (54)$$

and $y \in L^2(\Gamma^0)$.

Theorem 7 *The normal derivative of u_F on Γ^0 given by (40) (41) belongs to $L^2(\Gamma^0)$ and is the unique solution to (54).*

Proof. From Lemma 4, there remains only to prove uniqueness.
If $z \in (H^{\frac{1}{2}}(\Gamma^0))'$ is a solution to

$$\langle z - B^0 z, v\rangle = 0, \quad \forall v \in H^{\frac{1}{2}}(\Gamma^0), \quad \text{and} \quad \langle z, 1\rangle = 0, \tag{55}$$

then $z \in L^2(\Gamma^0)$. Let $\tilde{U}(z)$ be the harmonic extension of $U(z)$ in Ω^0. For $\sigma \in \mathcal{A}_n$, call e^σ the function of $\mathcal{V}(\Omega^0)$ defined by:

$$e^\sigma|_{\Omega^\sigma} = \tilde{U}(z) \circ \mathcal{M}_\sigma^{-1}(F_1, F_2), \quad \text{and} \quad e^\sigma|_{\Omega^0 \backslash \Omega^\sigma} = 0,$$

and call $u^{(n)}$ the function of $\mathcal{V}(\Omega^0)$ defined by

$$u^{(n)} = \sum_{p=0}^{n} \sum_{\sigma \in \mathcal{A}_p} e^\sigma.$$

From (52), (55) and (53), the function $u^{(n)}$ satisfies: for $v \in \mathcal{V}(\Omega^0)$

$$\int_{\Omega^0} \nabla u^{(n)} \cdot \nabla v + \frac{1}{2^{n+1}} \sum_{\sigma \in \mathcal{A}_{n+1}} \langle z, v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\rangle = 0. \tag{56}$$

But

$$\begin{aligned}
\left| \frac{1}{2^{n+1}} \sum_{\sigma \in \mathcal{A}_{n+1}} \langle z, v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\rangle \right| &\le \left(\frac{1}{2^{n+1}} \sum_{\sigma \in \mathcal{A}_{n+1}} \|v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\|_{L^2(\Gamma^0)} \right) \|z\|_{L^2(\Gamma^0)} \\
&\le \left(\frac{1}{2^{n+1}} \sum_{\sigma \in \mathcal{A}_{n+1}} \|v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\|^2_{L^2(\Gamma^0)} \right)^{\frac{1}{2}} \|z\|_{L^2(\Gamma^0)} \\
&= \|v|_{\Gamma^{n+1}}\|_{L^2(\Gamma^{n+1})} \|z\|_{L^2(\Gamma^0)}
\end{aligned}$$

From (15) in the case $q = 2$, we know that $\|v|_{\Gamma^{n+1}}\|_{L^2(\Gamma^{n+1})} \le C\|\nabla v\|L^2(\Omega^0)$. Therefore, the sequence $u^{(n)}$ is bounded in $\mathcal{V}(\Omega^0)$, and up to the extraction of a subsequence, we can assume that $u^{(n)}$ converges weakly to some function w in $\mathcal{V}(\Omega^0)$. It is clear that w coincides with $U(z)$ in ω^0.
Let $\mathcal{I}$ be an extension operator bounded from $W^{1,q}(\Omega^0)$ to $W^{1,q}(\widehat{\Omega}^0)$, for each number q, $1 < q < 2$. For any $v \in \mathcal{V}(\Omega^0)$,

$$\frac{1}{2^{n+1}} \sum_{\sigma \in \mathcal{A}_{n+1}} \langle z, v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\rangle = \sum_{\sigma \in \mathcal{A}_{n+1}} \int_{\Gamma^{n+1}} 1_{\Gamma^\sigma} \left(z \circ \mathcal{M}_\sigma^{-1}(F_1, F_2)\right) (\mathcal{I}v)|_{\Gamma^{n+1}}$$

Let $\bar{z}$ be the function defined on $\mathbb{R}$, periodic of period 6, and such that

$$\bar{z}(x_1) = \begin{cases} 0 & \text{if } x_1 \in (-3, -1) \\ z(x_1, 0) & \text{if } x_1 \in (-1, 1) \\ 0 & \text{if } x_1 \in (1, 3) \end{cases},$$

and let z_n be the function defined on $(-3,3)$ by $z_n(x_1) = \bar{z}(-3+2^n(x_1+3))$. The integral above can be written

$$\int_{-3}^{3} z_{n+1}(x_1)\widetilde{\mathcal{I}v}(x_1, y_{n+1})dx_1,$$

where y_{n+1} is the second coordinate of the points contained in Γ^{n+1}, and where $\widetilde{\mathcal{I}v}$ denotes the extension of $\mathcal{I}v$ by 0 out of $\widehat{\Omega^0}$. But it is easy to prove that, for $p \in \mathbb{R}$, $p \geq 1$,

$$\lim_{n\to\infty} \|\widetilde{\mathcal{I}v}(.,6) - \widetilde{\mathcal{I}v}(., y_{n+1})\|_{L^p(-3,3)} = 0.$$

On the other hand, we know that z_n converges weakly to 0 in $L^2(-3,3)$, since $\int_{-3}^{3} \bar{z} = 0$.
We have proved that, for all $v \in \mathcal{V}(\Omega^0)$,

$$\lim_{n\to\infty} \frac{1}{2^{n+1}} \sum_{\sigma\in\mathcal{A}_{n+1}} \langle z, v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)\rangle = 0. \tag{57}$$

This implies that w satisfies

$$\int_{\Omega^0} \nabla w \cdot \nabla v = 0, \quad \forall v \in \mathcal{V}(\Omega^0),$$

yielding that $w = 0$. Therefore $U(z) = 0$, which implies that $B^0 z = 0$ and finally that $z = 0$. Uniqueness is proved. ∎

Therefore, the normal derivative y of u_F is characterized by (54). Once y is known, the restriction of u_F to ω^0 is found by solving (49). Similarly, for any integer n, $n \geq 1$, the restriction of u_F to ω^n can be found by solving the variational problem: $u_F|_{\omega^n} \in \mathcal{V}(\omega^n)$ and for all $v \in \mathcal{V}(\omega^n)$,

$$\begin{aligned}&\int_{\omega^n} \nabla u_F \cdot \nabla v + \sum_{\sigma\in\mathcal{A}_{n+1}} \left\langle T^0\left(u_F \circ \mathcal{M}_\sigma(F_1,F_2)\right), v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)\right\rangle \\ &= -\frac{1}{2^{n+1}} \sum_{\sigma\in\mathcal{A}_{n+1}} \langle y, v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1,F_2)\rangle .\end{aligned} \tag{58}$$

We see that the knowledge of the operator T^0 enables to compute exactly the restriction of u_F to any domain ω^n, $n \geq 0$.
Note also that u_F can be computed by the following induction on n: for all $n \geq 1$ and $\sigma \in \mathcal{A}_n$, $u_F|_{\mathcal{M}_\sigma(F_1,F_2)(\omega^0)} \in H^1(\mathcal{M}_\sigma(F_1,F_2)(\omega^0))$,

- $u_F|_{\mathcal{M}_\sigma(F_1,F_2)(\Gamma^0)}$ in known from step $n-1$,
- $\forall v \in H^1(\mathcal{M}_\sigma(F_1,F_2)(\omega^0))$ such that $v|_{\mathcal{M}_\sigma(F_1,F_2)(\Gamma^0)} = 0$,

$$\begin{aligned}&\int_{\mathcal{M}_\sigma(F_1,F_2)(\omega^0)} \nabla u_F \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0\left(u_F \circ \mathcal{M}_\sigma(F_1,F_2)\circ F_i\right), v|_{\mathcal{M}_\sigma(F_1,F_2)\circ F_i(\Gamma^0)} \circ \mathcal{M}_\sigma(F_1,F_2)\circ F_i\right\rangle \\ &= -\frac{1}{2^{n+1}} \sum_{i=1}^{2} \left\langle y, v|_{\mathcal{M}_\sigma(F_1,F_2)\circ F_i(\Gamma^0)} \circ \mathcal{M}_\sigma(F_1,F_2)\circ F_i\right\rangle .\end{aligned}$$

Note that the domains involved by the induction above are all deduced from ω^0 by affine mappings. Therefore, when implementing the discrete analogue of this induction by using the finite element method, no other mesh than that of ω^0 will be needed.

4.4 Solving (26) with g being a Haar wavelet

4.4.1 g is the Haar mother wavelet

We start by considering the boundary problem (26) with $g = g^0 = 1_{\{x_1<0\}} - 1_{\{x_1>0\}}$, ($g^0$ is the Haar mother wavelet), namely to find $u^0 \in \mathcal{V}(\Omega^0)$ such that for all $v \in \mathcal{V}(\Omega^0)$,

$$\int_{\Omega^0} \nabla u^0 \cdot \nabla v = \frac{1}{3} \int_{\Gamma^\infty} \left(1_{\{x_1<0\}} - 1_{\{x_1>0\}}\right) v. \tag{59}$$

Consider the function $\tilde{u}^0 \in \mathcal{V}(\Omega^0)$:

$$\begin{array}{rcl} \tilde{u}^0|_{\omega^0} & = & 0, \\ \tilde{u}^0|_{F_i(\Omega^0)} & = & \frac{(-1)^{i+1}}{2} u_F \circ F_i^{-1}, \quad i = 1, 2 \end{array} \tag{60}$$

and the error $e = u^0 - \tilde{u}^0$ satisfies, for all $v \in \mathcal{V}(\Omega^0)$,

$$\int_{\Omega^0} \nabla e \cdot \nabla v = -\frac{1}{2} \sum_{i=1}^{2} (-1)^{i+1} \left\langle y, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle, \tag{61}$$

where y is defined in (41).
From the fact that e and u^0 coincide in ω^0, we obtain that $u^0|_{\omega^0}$ is characterized by: for $v \in \mathcal{V}(\omega^0)$,

$$\int_{\omega^0} \nabla u^0 \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0 \left(u_F \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle = -\frac{1}{2} \sum_{i=1}^{2} (-1)^{i+1} \left\langle y, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle. \tag{62}$$

Once $u^0|_{\omega^0}$ is known, then

$$u^0|_{F_i(\Omega_0)} = \frac{(-1)^{i+1}}{2} \tilde{u}_F \circ F_i^{-1} + \left(\mathcal{H}^0(u^0|_{F_i(\Gamma_0)} \circ F_i)\right) \circ F_i^{-1},$$

so u^0 can be computed by an induction formula similar to the one for u_F.
It will be useful to denote by y^0 the normal derivative of u^0 on Γ^0. We have $\int_{\Gamma^0} y^0 = 0$.

4.4.2 g is a wavelet in the level n, $n > 0$

Let n be a positive integer and take $\sigma \in \mathcal{A}_n$. Call g^σ the Haar wavelet on Γ^∞, defined by

$$\begin{array}{r} g^\sigma|_{\mathcal{M}_\sigma(F_1,F_2)(\Gamma^\infty)} = g^0 \circ \mathcal{M}_\sigma^{-1}(F_1, F_2), \\ g^\sigma|_{\Gamma^\infty \setminus \mathcal{M}_\sigma(F_1,F_2)(\Gamma^\infty)} = 0, \end{array} \tag{63}$$

and call u^σ the function in $\mathcal{V}(\Omega^0)$ such that for all $v \in \mathcal{V}(\Omega^0)$,

$$\int_{\Omega^0} \nabla u^\sigma \cdot \nabla v = \frac{1}{3} \int_{\Gamma^\infty} g^\sigma v, \tag{64}$$

and y^σ the normal derivative of u^σ on Γ^0.
Define $\tilde{u}^\sigma \in \mathcal{V}(\Omega^0)$ by

$$\tilde{u}^\sigma|_{\Omega^\sigma} = \frac{u^0 \circ \mathcal{M}_\sigma^{-1}(F_1, F_2)}{2^n}, \quad \text{and} \quad \tilde{u}^\sigma|_{\Omega^0 \setminus \Omega^\sigma} = 0.$$

The error $e = u^\sigma - \tilde{u}^\sigma$ satisfies, for all $v \in \mathcal{V}(\Omega^0)$,

$$\int_{\Omega^0} \nabla e \cdot \nabla v = -\frac{1}{2^n} \left\langle y^0, v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2) \right\rangle. \tag{65}$$

The restriction of e to ω^{n-1}, i.e. $u^\sigma|_{\omega^{n-1}}$ satisfies: for all $v \in \mathcal{V}(\omega^{n-1})$,

$$\int_{\omega^{n-1}} \nabla u^\sigma \cdot \nabla v + \sum_{\mu \in \mathcal{A}_n} \left\langle T^0 \left(u^\sigma \circ \mathcal{M}_\mu(F_1, F_2)\right), v|_{\Gamma^\mu} \circ \mathcal{M}_\mu(F_1, F_2) \right\rangle = -\frac{1}{2^n} \left\langle y^0, v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2) \right\rangle. \tag{66}$$

It is possible to compute the family (u^σ, y^σ) by an induction on n, from the following observation: if $\sigma = F_i \circ \eta$ for some $i \in \{1, 2\}$ and $\eta \in \mathcal{A}_{n-1}$, then (u^σ, y^σ) can be computed from (u^η, y^η), by solving, for all $v \in \mathcal{V}(\omega^0)$,

$$\int_{\omega^0} \nabla u^\sigma|_{\omega^0} \cdot \nabla v + \sum_{\mu \in \mathcal{A}_1} \left\langle T^0 \left(u^\sigma|_{\Gamma^\mu} \circ \mathcal{M}_\mu(F_1, F_2)\right), v|_{\Gamma^\mu} \circ \mathcal{M}_\mu(F_1, F_2) \right\rangle = -\frac{1}{2} \left\langle y^\eta, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle,$$

with

$$\begin{aligned} u^\sigma|_{F_i(\Omega^0)} &= \frac{1}{2} u^\eta \circ F_i^{-1} + \left(\mathcal{H}^0(u^\sigma|_{F_i(\Gamma_0)} \circ F_i)\right) \circ F_i^{-1}, \\ u^\sigma|_{F_j(\Omega^0)} &= \Big(\mathcal{H}^0(u^\sigma|_{F_j(\Gamma_0)} \circ F_j)\Big) \circ F_j^{-1} \quad \text{with } j = 1 - i. \end{aligned}$$

and $\forall v \in H^1(\omega^0)$,

$$\langle y^\sigma, v \rangle = \int_{\omega^0} \nabla u^\sigma|_{\omega^0} \cdot \nabla v + \sum_{\mu \in \mathcal{A}_1} \left\langle T^0 \left(u^\sigma|_{\Gamma^\mu} \circ \mathcal{M}_\mu(F_1, F_2)\right), v|_{\Gamma^\mu} \circ \mathcal{M}_\mu(F_1, F_2) \right\rangle + \frac{1}{2} \left\langle y^\eta, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle.$$

4.4.3 The asymptotic behavior of $u^\sigma|_{\omega^p}$, for $\sigma \in \mathcal{A}_n$ as $n \to \infty$

For what follows, it is useful to define the spaces

$$\begin{aligned} W^0 &= \{g \in (H^{\frac{1}{2}}(\Gamma^0))' \text{ such that } \langle g, 1 \rangle = 0\}, \\ W^\sigma &= \{g \in (H^{\frac{1}{2}}(\Gamma^\sigma))' \text{ such that } \langle g, 1 \rangle = 0\}, \qquad \text{for } \sigma \in \mathcal{A}_n, n > 0. \end{aligned} \tag{67}$$

It is clear that

$$\begin{aligned} \|g\|_{W^0} &= \sup_{v \in H^1(\Omega^0), \|\nabla v\|_{L^2(\Omega^0)} \neq 0} \frac{\langle g, v|_{\Gamma^0} \rangle}{\|\nabla v\|_{L^2(\Omega^0)}}, \qquad \forall g \in W^0, \\ \|g\|_{W^\sigma} &= \sup_{v \in H^1(\Omega^\sigma), \|\nabla v\|_{L^2(\Omega^\sigma)} \neq 0} \frac{\langle g, v \rangle}{\|\nabla v\|_{L^2(\Omega^\sigma)}}, \qquad \forall g \in W^\sigma, \end{aligned} \tag{68}$$

are respectively norms on W^0 and W^σ.

Remark 3 *For $\mu \in \mathcal{A}_n$ and $\phi^\mu \in W^\mu$, defining the distribution $\phi^0 \in W^0$ by $\forall v \in H^{\frac{1}{2}}(\Gamma^0)$, $\langle \phi^0, v \rangle = \langle \phi^\mu, v \circ \mathcal{M}_\mu^{-1}(F_1, F_2) \rangle$, we have that $\|\phi^0\|_{W^0} = \|\phi^\mu\|_{W^\mu}$.*

Lemma 5 *For all $g \in W^0$,*

$$\|g\|_{W^0} = \sup_{\substack{v \in H^{\frac{1}{2}}(\Gamma^0), \\ v \text{ nonconstant}}} \frac{\langle g, v \rangle}{\|\nabla \mathcal{H}^0(v)\|_{L^2(\Omega^0)}}. \tag{69}$$

Proof. Clearly, for all nonconstant $v \in H^{\frac{1}{2}}(\Gamma^0)$, and for all lifting $\psi \in H^1(\Omega^0)$ of v, (i.e. such that $\psi|_{\Gamma^0} = v$), $\frac{\langle g,v\rangle}{\|\nabla\psi\|_{L^2(\Omega^0)}} \le \|g\|_{W^0}$. Therefore, $\frac{\langle g,v\rangle}{\|\nabla\mathcal{H}^0(v)\|_{L^2(\Omega^0)}} \le \|g\|_{W^0}$, which yields

$$\|g\|_{W^0} \ge \sup_{\substack{v \in H^{\frac{1}{2}}(\Gamma^0),\\ v \text{ nonconstant}}} \frac{\langle g,v\rangle}{\|\nabla\mathcal{H}^0(v)\|_{L^2(\Omega^0)}}.$$

On the other hand, for all nonconstant $v \in H^1(\Omega^0)$, we have

- either $v|_{\Gamma^0}$ is constant, thus $\langle g, v|_{\Gamma^0}\rangle = 0$ and
$$\frac{|\langle g, v|_{\Gamma^0}\rangle|}{\|\nabla v\|_{L^2(\Omega^0)}} = 0 \le \sup_{\substack{w \in H^{\frac{1}{2}}(\Gamma^0),\\ w \text{ nonconstant}}} \frac{|\langle g,w\rangle|}{\|\nabla\mathcal{H}^0(w)\|_{L^2(\Omega^0)}}. \tag{70}$$

- or $v|_{\Gamma^0}$ is not constant: in this case, we use the fact that
$$\|\nabla v\|_{L^2(\Omega^0)} \ge \|\nabla\mathcal{H}^0(v|_{\Gamma^0})\|_{L^2(\Omega^0)} > 0,$$
because the harmonic lifting is the one with minimal energy. This implies
$$\frac{|\langle g, v|_{\Gamma^0}\rangle|}{\|\nabla v\|_{L^2(\Omega^0)}} \le \frac{|\langle g, v|_{\Gamma^0}\rangle|}{\|\nabla\mathcal{H}^0(v|_{\Gamma^0})\|_{L^2(\Omega^0)}}.$$
Therefore,
$$\begin{aligned}\sup_{\substack{v \in H^1(\Omega^0),\\ \|\nabla v\|_{L^2(\Omega^0)} \neq 0}} \frac{|\langle g, v|_{\Gamma^0}\rangle|}{\|\nabla v\|_{L^2(\Omega^0)}} &\le \sup_{\substack{v \in H^1(\Omega^0),\\ \|\nabla v\|_{L^2(\Omega^0)} \neq 0}} \frac{|\langle g, v|_{\Gamma^0}\rangle|}{\|\nabla\mathcal{H}^0(v|_{\Gamma^0})\|_{L^2(\Omega^0)}}\\ &\le \sup_{\substack{v \in H^{\frac{1}{2}}(\Gamma^0),\\ v \text{ nonconstant}}} \frac{\langle g,v\rangle}{\|\nabla\mathcal{H}^0(v)\|_{L^2(\Omega^0)}}.\end{aligned}$$

Finally, for all $g \in W^0$,

$$\|g\|_{W^0} \le \sup_{\substack{v \in H^{\frac{1}{2}}(\Gamma^0),\\ v \text{ nonconstant}}} \frac{\langle g,v\rangle}{\|\nabla\mathcal{H}^0(v)\|_{L^2(\Omega^0)}}.$$

■

Lemma 6 *There exists a positive constant C such that u^σ defined by (64) satisfies*

$$\|\nabla u^\sigma\|_{L^2(\Omega^0)} \le \frac{C}{2^n}. \tag{71}$$

Proof. From (65) and (68), and since $\int_{\Gamma^0} y_0 = 0$, we deduce that

$$\begin{array}{rcl} \|\nabla e\|^2_{L^2(\Omega^0)} & \le & \frac{1}{2^n}\|y^0\|_{W^0}\|\nabla(e\circ\mathcal{M}_\sigma(F_1,F_2))\|_{L^2(\Omega^0)} \\ & = & \frac{1}{2^n}\|y^0\|_{W^0}\|\nabla e\|_{L^2(\Omega^\sigma)}, \end{array} \tag{72}$$

which yields that

$$\|\nabla e\|_{L^2(\Omega^0)} \lesssim \frac{1}{2^n}. \tag{73}$$

On the other hand,

$$\|\nabla\tilde{u}^\sigma\|_{L^2(\Omega^0)} = \frac{1}{2^n}\|\nabla(u^0\circ\mathcal{M}_\sigma^{-1}(F_1,F_2))\|_{L^2(\Omega^\sigma)} = \frac{1}{2^n}\|\nabla u^0\|_{L^2(\Omega^0)}. \tag{74}$$

The desired result follows from (73) and (74). ■

Lemma 7 *Consider $i\in\{1,2\}$ and $y\in W^0$. The unique function $f\in\mathcal{V}(\Omega^0)$ such that, for all $v\in\mathcal{V}(\Omega^0)$,*

$$\int_{\Omega^0}\nabla f\cdot\nabla v = \langle y, v|_{F_i(\Gamma^0)}\circ F_i\rangle \tag{75}$$

satisfies $\frac{\partial f}{\partial n}|_{\Gamma^0}\in W^0$ and

$$\|\frac{\partial f}{\partial n}|_{\Gamma^0}\|_{W^0} \le \rho\|y\|_{W^0}, \tag{76}$$

where ρ, $0\le\rho<1$ is the constant appearing in (33).

Proof. We know that for all $v\in H^1(\Omega^0)$

$$\left\langle\frac{\partial f}{\partial n}|_{\Gamma^0}, v|_{\Gamma^0}\right\rangle = \int_{\Omega^0}\nabla f\cdot\nabla v - \langle y, v|_{F_i(\Gamma^0)}\circ F_i\rangle.$$

Therefore, for all $v\in H^{\frac{1}{2}}(\Gamma^0)$,

$$\left\langle\frac{\partial f}{\partial n}|_{\Gamma^0}, v|_{\Gamma^0}\right\rangle = \int_{\Omega^0}\nabla f\cdot\nabla\mathcal{H}^0(v) - \langle y, (\mathcal{H}^0(v))|_{F_i(\Gamma^0)}\circ F_i\rangle. \tag{77}$$

But from the definition of $\mathcal{H}^0(v)$, $\int_{\Omega^0}\nabla f\cdot\nabla\mathcal{H}^0(v)=0$. Therefore, from (69),

$$\begin{aligned} \left\langle\frac{\partial f}{\partial n}|_{\Gamma^0}, v|_{\Gamma^0}\right\rangle &= -\langle y, (\mathcal{H}^0(v))|_{F_i(\Gamma^0)}\circ F_i\rangle \\ &\le \|y\|_{W^0}\|\nabla(\mathcal{H}^0(v)\circ F_i)\|_{L^2(\Omega^0)} = \|y\|_{W^0}\|\nabla\mathcal{H}^0(v)\|_{L^2(F_i(\Omega^0))}. \end{aligned} \tag{78}$$

But from (33),

$$\|\nabla\mathcal{H}^0(v)\|_{L^2(F_i(\Omega^0))} \le \rho\|\nabla\mathcal{H}^0(v)\|_{L^2(\Omega^0)}.$$

This and (69) yield the desired result. ■

Lemma 8 *For all $n\ge 1$, for all $\sigma\in\mathcal{A}_n$ and for all $y\in W^0$, the unique function $f\in\mathcal{V}(\Omega^0)$ such that for all $v\in\mathcal{V}(\Omega^0)$,*

$$\int_{\Omega^0}\nabla f\cdot\nabla v = \langle y, v|_{\Gamma^\sigma}\circ\mathcal{M}_\sigma(F_1,F_2)\rangle, \tag{79}$$

satisfies, for all integer $p\ge 1$ such that $n-p\ge 0$,

$$\|\nabla f\|_{L^2(\omega^{n-p})} \le \rho^{p-1}\|y\|_{W^0}, \tag{80}$$

where $\rho<1$ is the constant introduced in (33).

Proof. The proof is done by induction on n.
The result is true for $n = 1$, since $\|\nabla f\|_{L^2(\Omega^0)} \le \|y\|_{W^0}$.
Assume that the induction property is true up to $n-1$, $n > 1$. For $\sigma \in \mathcal{A}_n$, let $\tilde{f} \in \mathcal{V}(\Omega^0)$ be defined by: $\tilde{f}|_{\omega^{n-2}} = 0$ and for all $v \in \mathcal{V}(\Omega^{n-1})$,

$$\int_{\Omega^{n-1}} \nabla \tilde{f} \cdot \nabla v = \langle y, v|_{\Gamma^\sigma} \circ \mathcal{M}_\sigma(F_1, F_2)\rangle. \tag{81}$$

Call μ the unique element of $\mathcal{A}_{n-1}$ such that $\Gamma^\sigma \subset \Omega^\mu$. It is clear that $\tilde{f}$ vanishes out of Ω^μ, and that
$\frac{\partial \tilde{f}}{\partial n}|_{\Gamma^\mu} \in W^\mu$ (here n is the unit vector normal to Γ^μ pointing outside Ω^μ). Lemma 7 applied to the function $\tilde{f} \circ \mathcal{M}_\mu(F_1, F_2) \in \mathcal{V}(\Omega^0)$ and Remark 3 applied to $\phi^\mu = \frac{\partial \tilde{f}}{\partial n}|_{\Gamma^\mu}$ tell us that

$$\|\frac{\partial \tilde{f}}{\partial n}|_{\Gamma^\mu}\|_{W^\mu} \le \rho \|y\|_{W^0}. \tag{82}$$

Consider now the function $\hat{f} = f - \tilde{f}$, where f is defined by (79). We have that

$$\begin{array}{l} \hat{f}|_{\Omega^0 \setminus \Omega^\mu} = f|_{\Omega^0 \setminus \Omega^\mu}, \\ \int_{\Omega^0} \nabla \hat{f} \cdot \nabla v = \langle \frac{\partial \tilde{f}}{\partial n}|_{\Gamma^\mu}, v\rangle, \quad \forall v \in \mathcal{V}(\Omega^0). \end{array} \tag{83}$$

The induction hypothesis together with (83) and Remark 3 tell us that for all integer $q \ge 1$ such that $n - 1 - q \ge 0$,

$$\|\nabla \hat{f}\|_{L^2(\omega^{n-1-q})} \le \rho^{q-1} \|\frac{\partial \tilde{f}}{\partial n}|_{\Gamma^\mu}\|_{W^\mu}. \tag{84}$$

From (82) and (84), taking $p = q + 1$, we deduce that for all $p \ge 2$ such that $n - p \ge 0$,

$$\|\nabla f\|_{L^2(\omega^{n-p})} \le \rho^{p-1} \|y\|_{W^0}.$$

The case $p = 1$ comes from the fact that $\|\nabla f\|_{L^2(\Omega^0)} \le \|y\|_{W^0}$. The induction property is proved at step n. ■
Going back to the function u^σ defined by (64), using (65) and Lemma 8, we have proved the

Theorem 8 *There exists a constant C such that for all nonnegative integers p, n such that $0 \le p < n - 1$, the function u^σ defined by (64) satisfies*

$$\|\nabla u^\sigma\|_{L^2(\omega^p)} \le \frac{C \rho^{n-p}}{2^n}, \tag{85}$$

where $\rho < 1$ is the constant appearing in (33).

4.5 Approximate solutions to (26) for general data g

We are now aiming at approximating the solution to (26) where $g \in L^2(\Gamma^\infty)$. The idea is to express g in the Haar wavelet basis:

$$g = \alpha_F 1_{\Gamma^\infty} + \alpha_0 g^0 + \sum_{n=1}^{\infty} \sum_{\sigma \in \mathcal{A}_n} \alpha_\sigma g^\sigma, \tag{86}$$

where g^0 is defined in § 4.4.1 and g^σ is defined in § 4.4.2, or in equivalent manner,

$$\lim_{N \to \infty} \left\| g - \alpha_F 1_{\Gamma^\infty} - \alpha_0 g^0 - \sum_{n=1}^{N} \sum_{\sigma \in \mathcal{A}_n} \alpha_\sigma g^\sigma \right\|_{L^2(\Gamma^\infty)} = 0. \tag{87}$$

Note that $\alpha_F = \langle g \rangle_{\Gamma^\infty}$. We have the

Proposition 3 *The function u solution to (26) satisfies*

$$\lim_{N\to\infty} \left\| u - \alpha_F u_F - \alpha_0 u^0 - \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{A}_n} \alpha_\sigma u^\sigma \right\|_{H^1(\Omega^0)} = 0. \tag{88}$$

where the functions u_F, u^0 and u^σ are respectively defined in (40), (59) and (64).
Furthermore, there exists a constant C such that such that for all integers p, N, with $0 \le p < N-1$,

$$\left\| u|_{\omega^p} - \alpha_F u_F|_{\omega^p} - \alpha_0 u^0|_{\omega^p} - \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{A}_n} \alpha_\sigma u^\sigma|_{\omega^p} \right\|_{H^1(\omega^p)} \le \frac{C\rho^{N-p}}{\sqrt{2^N}} \|g\|_{L^2(\Gamma^\infty)}, \tag{89}$$

where $\rho < 1$ is the constant appearing in (33).

Proof. The convergence result (88) stems from (87) and the stability properties of problem (26).
From (85), we see that

$$\begin{aligned}
\| \sum_{n=N+1}^{\infty} \sum_{\sigma\in\mathcal{A}_n} \alpha_\sigma u^\sigma|_{\omega^p} \|^2_{H^1(\omega^p)} &\le \left(\sum_{n=N+1}^{\infty} \sum_{\sigma\in\mathcal{A}_n} \|u^\sigma|_{\omega^p}\|^2_{H^1(\omega^p)} \right) \left(\sum_{n=N+1}^{\infty} \sum_{\sigma\in\mathcal{A}_n} |\alpha_\sigma|^2 \right) \\
&\le C^2 \left(\sum_{n=N+1}^{\infty} \sum_{\sigma\in\mathcal{A}_n} \frac{\rho^{2(n-p)}}{4^n} \right) \|g\|^2_{L^2(\Gamma^\infty)} \\
&= C^2 \left(\sum_{n=N+1}^{\infty} \frac{\rho^{2(n-p)}}{2^n} \right) \|g\|^2_{L^2(\Gamma^\infty)} \\
&\lesssim \frac{\rho^{2(N-p)}}{2^N} \|g\|^2_{L^2(\Gamma^\infty)},
\end{aligned}$$

which is the desired estimate ■

Remark 4 *If g is more regular, for example $g \in W^{1,s}(\Gamma^\infty)$ then the quantity*

$$\left\| g - \alpha_F 1_{\Gamma^\infty} - \alpha_0 g^0 - \sum_{n=1}^{N} \sum_{\sigma\in\mathcal{A}_n} \alpha_\sigma g^\sigma \right\|_{L^2(\Gamma^\infty)}$$

converges to 0 as a power of $\frac{1}{N}$, and the error estimate (89) is improved.

Proposition 3 says that the restriction to ω^p of the solution u to (26) can be computed accurately by truncating the expansion in (88) to a relatively small order. In order words, the restriction of u to ω^p is not affected by the highly oscillating components of g.

5 A self similar finite element method

5.1 Function spaces

We consider a regular family of triangulations $\mathcal{T}_h^0$ of ω^0, (see [5]) with the special property that for $i = 1, 2$, the set of the nodes of $\mathcal{T}_h^0$ lying on Γ_i^1 is the image by F_i of the set the nodes of $\mathcal{T}_h^0$ lying on Γ^0. Thanks to this property, the set of triangles $\mathcal{T}_h^1 = \mathcal{T}_h^0 \cup \bigcup_{i=1}^2 F_i(\mathcal{T}_h^0)$ is a triangulation of ω^1. Similarly, let us call $\mathcal{T}_h^N$ the triangulation of ω^N: $\mathcal{T}_h^N = \bigcup_{n=0}^N \cup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(F_1, F_2)(\mathcal{T}_h^0)$.

Finally, it is possible to construct a self-similar mesh $\mathcal{T}_h^\infty = \bigcup_{n=0}^\infty \cup_{\sigma\in\mathcal{A}_n} \mathcal{M}_\sigma(F_1, F_2)(\mathcal{T}_h^0)$. Let us call $V_h(\omega^n)$ the set of piecewise linear functions:

$$V_h(\omega^n) = \{v_h \in \mathcal{C}^0(\overline{\omega^n}),\ \forall\tau \in \mathcal{T}_h^n,\ v_h|_\tau \text{ is linear}\}. \tag{90}$$

Similarly,

$$V_h(\Omega^0) = \{v_h \in H^1(\Omega^0),\ \forall\tau \in \mathcal{T}_h^\infty,\ v_h|_\tau \text{ is linear}\}. \tag{91}$$

It is clear that for all $v_h \in V_h(\Omega^0)$, the restriction of v_h to ω^n belongs to $V_h(\omega^n)$. Consider $V_{h,0}(\Omega^0) = \{v_h \in V_h(\Omega^0);\ v_h|_{\Gamma^0} = 0\}$, and let $V_h(\Gamma^n)$ (resp., $V_h(\Gamma_i^n)$, $1 \le i \le 2^N$) be the space of the traces of the functions of $V_h(\Omega^0)$ on Γ^n (resp., Γ_i^n). It is clear that for all $v_h \in V_h(\Gamma_i^1)$, $v_h \circ F_i \in V_h(\Gamma^0)$.
We have the approximation result, whose proof is skipped for brevity:

Lemma 9 *For all $u \in H^1(\Omega^0)$,*

$$\lim_{h\to 0} \inf_{u_h \in V_h(\Omega^0)} \|u - u_h\|_{H^1(\Omega^0)} = 0.$$

5.2 The discrete Dirichlet to Neumann operator

The results given in this section are proved in [1].

5.2.1 Discrete harmonic lifting and Dirichlet to Neumann operator

We are now ready to define the discrete harmonic lifting operator $\mathcal{H}_h^0 : V_h(\Gamma^0) \mapsto V_h(\Omega^0)$, $\forall u_h \in V_h(\Gamma^0)$, $\mathcal{H}_h^0(u_h)|_{\Gamma^0} = u_h$ and

$$\forall v_h \in V_{h,0}(\Omega^0), \qquad \int_{\Omega^0} \nabla\mathcal{H}_h^0(u_h) \cdot \nabla v_h = 0. \tag{92}$$

We have the analogue of Theorem 5:

Theorem 9 *There exists a constant $\rho < 1$, independent of h such that for all $u_h \in V_h(\Gamma^0)$,*

$$\int_{\Omega^N} |\nabla\mathcal{H}_h^0(u_h)|^2 \le \rho^N \int_{\Omega^0} |\nabla\mathcal{H}_h^0(u_h)|^2. \tag{93}$$

We can also define the discrete Dirichlet-Neumann operator $T_h^0 : V_h(\Gamma^0) \mapsto \big(V_h(\Gamma^0)\big)'$

$$\langle T_h^0 u_h, v_h\rangle = \int_{\Omega^0} \nabla\mathcal{H}_h^0(u_h) \cdot \nabla\mathcal{H}_h^0(v_h) = \int_{\Omega^0} \nabla\mathcal{H}^0(u_h) \cdot \nabla\tilde{v}_h, \tag{94}$$

for any function $\tilde{v}_h \in V_h(\Omega^0)$ such that $\tilde{v}_h|_{\Gamma^0} = v_h$. Exactly as for the continuous problem, we introduce the cone $\mathbb{O}_h$ of self adjoint, positive semi-definite, bounded linear operators from $V_h(\Gamma^0)$ to its dual, vanishing on the constants, and the mapping $\mathbb{M}_h : \mathbb{O}_h \mapsto \mathbb{O}_h$ defined as follows: for $Z_h \in \mathbb{O}_h$, define $\mathbb{M}_h(Z_h)$ by $\forall u_h \in V_h(\Gamma^0)$, $\forall v_h \in V_h(\omega^0)$,

$$\langle \mathbb{M}_h(Z_h)u_h, v_h|_{\Gamma^0}\rangle = \int_{\omega^0} \nabla\hat{u}_h \cdot \nabla v_h + \sum_{i=1}^2 \left\langle Z_h(\hat{u}_h|_{\Gamma_i^1} \circ F_i), v_h|_{\Gamma_i^1} \circ F_i \right\rangle, \tag{95}$$

where $\hat{u}_h \in V_h(\omega^0)$ is such that $\hat{u}_h|_{\Gamma^0} = u_h$ and

$$\forall v_h \in V_h(\omega^0) \text{ with } v_h|_{\Gamma_0} = 0, \qquad \int_{\omega^0} \nabla\hat{u}_h \cdot \nabla v_h + \sum_{i=1}^2 \left\langle Z_h(\hat{u}_h|_{\Gamma_i^1} \circ F_i), v_h|_{\Gamma_i^1} \circ F_i \right\rangle = 0. \tag{96}$$

We have the analogue of Theorem 6:

Theorem 10 *The operator T_h^0 is the unique fixed point of $\mathbb{M}_h$ and for all $Z_h \in \mathbb{O}_h$, there exists a positive constant C independent of n such that, for all $n \geq 0$,*

$$\|\mathbb{M}_h^n(Z_h) - T_h^0\| \leq C\rho^{\frac{n}{4}}, \tag{97}$$

where ρ, $0 < \rho < 1$ is the constant appearing in Theorem 9.

5.2.2 The linear algebra viewpoint

Let us call $N_h(\omega^0)$ (resp., N) the dimension of $V_h(\omega^0)$, (resp., $V_h(\Gamma^0)$). Call $(x_i)_{i=1,\dots,N}$ the abscissa of the mesh-nodes lying on Γ^0, ordered increasingly. Let us introduce the nodal basis $(\phi_i)_{i=1,\dots,N_h(\omega^0)}$ of $V_h(\omega^0)$ ordered as follows:

1. for $j = 1, \dots, N$, ϕ_j corresponds to the node $(x_j, 0) \in \Gamma^0$.
2. for $i = 1, 2$ and $j = 1, \dots, N$ ϕ_{iN+j} corresponds to the node $F_i(x_j, 0) \in \Gamma_i^1$.
3. for $3N < j \leq N_h(\omega^0)$, the node corresponding to ϕ_j belongs to $\overline{\omega_0} \backslash (\Gamma^0 \cup \Gamma^1)$.

Consider the bilinear for $a_h : V_h(\omega^0) \times V_h(\omega^0) \mapsto \mathbb{R}$: $a_h(u_h, v_h) = \int_{\omega^0} \nabla u_h \cdot \nabla v_h$, and let A be the matrix of a_h in the nodal basis described above. We have the block decomposition

$$A = \begin{pmatrix} A_{\Gamma^0,\Gamma^0} & 0 & A_{\Gamma^0,I} \\ 0 & A_{\Gamma^1,\Gamma^1} & A_{\Gamma^1,I} \\ A_{\Gamma^0,I}^T & A_{\Gamma^1,I}^T & A_{I,I} \end{pmatrix}, \qquad \begin{array}{l} A_{\Gamma^0,\Gamma^0} \in \mathbb{R}^{N\times N} \\ A_{\Gamma^1,\Gamma^1} \in \mathbb{R}^{2N\times 2N} \end{array}. \tag{98}$$

The block $A_{I,I}$ is positive definite; it is the matrix arising when dealing with a Poisson problem with Dirichlet conditions on $\Gamma^0 \cup \Gamma^1$ and Neumann conditions on Σ^0. The Schur complement of A obtained by eliminating the degrees of freedom corresponding to the mesh nodes in $\omega^0 \cup \Sigma^0$ is $S \in R^{3N\times 3N}$:

$$S = \begin{pmatrix} S_{\Gamma^0,\Gamma^0} & S_{\Gamma^0,\Gamma^1} \\ S_{\Gamma^0,\Gamma^1}^T & S_{\Gamma^1,\Gamma^1} \end{pmatrix}, \qquad \begin{array}{l} S_{\Gamma^0,\Gamma^0} = A_{\Gamma^0,\Gamma^0} - A_{\Gamma^0,I}A_{I,I}^{-1}A_{\Gamma^0,I}^T \in \mathbb{R}^{N\times N} \\ S_{\Gamma^1,\Gamma^1} = A_{\Gamma^1,\Gamma^1} - A_{\Gamma^1,I}A_{I,I}^{-1}A_{\Gamma^1,I}^T \in \mathbb{R}^{2N\times 2N} \\ S_{\Gamma^0,\Gamma^1} = -A_{\Gamma^0,I}A_{I,I}^{-1}A_{\Gamma^1,I}^T \in \mathbb{R}^{N\times 2N} \end{array}. \tag{99}$$

The block S_{Γ^0,Γ^0} is the matrix in the nodal basis of $V_h(\Gamma^0)$ of the bilinear form mapping $(u_h, v_h) \in V_h(\Gamma^0) \times V_h(\Gamma^0)$ to $\int_{\omega^0} \nabla \hat{u}_h \nabla \hat{v}_h$, where $\hat{u}_h$ and $\hat{v}_h$ satisfy

$$\begin{array}{l} \hat{u}_h \in V_h(\omega^0), \quad \hat{u}_h|_{\Gamma_0} = u_h, \quad \hat{u}_h|_{\Gamma_1} = 0, \\ \hat{v}_h \in V_h(\omega^0), \quad \hat{v}_h|_{\Gamma_0} = v_h, \quad \hat{v}_h|_{\Gamma_1} = 0, \\ \forall w_h \in V_h(\omega^0) \text{ such that } w_h|_{\Gamma_0} = 0, \quad w_h|_{\Gamma_1} = 0, \quad \int_{\omega^0} \nabla \hat{u}_h \nabla w_h = \int_{\omega^0} \nabla \hat{v}_h \nabla w_h = 0. \end{array}$$

The block S_{Γ^1,Γ^1} is the matrix in the nodal basis of $V_h(\Gamma^1)$ of the bilinear form mapping $(u_h, v_h) \in V_h(\Gamma^1) \times V_h(\Gamma^1)$ to $\int_{\omega^0} \nabla \hat{u}_h \nabla \hat{v}_h$, where where $\hat{u}_h$ and $\hat{v}_h$ satisfy

$$\begin{array}{l} \hat{u}_h \in V_h(\omega^0), \quad \hat{u}_h|_{\Gamma_1} = u_h, \quad \hat{u}_h|_{\Gamma_0} = 0, \\ \hat{v}_h \in V_h(\omega^0), \quad \hat{v}_h|_{\Gamma_1} = v_h, \quad \hat{v}_h|_{\Gamma_0} = 0, \\ \forall w_h \in V_h(\omega^0) \text{ such that } w_h|_{\Gamma_1} = 0, \quad w_h|_{\Gamma_0} = 0, \quad \int_{\omega^0} \nabla \hat{u}_h \nabla w_h = \int_{\omega^0} \nabla \hat{v}_h \nabla w_h = 0. \end{array}$$

The block S_{Γ^0,Γ^1} is the matrix of the bilinear form mapping $(u_h, v_h) \in V_h(\Gamma^0) \times V_h(\Gamma^1)$ to $\int_{\omega^0} \nabla \hat{u}_h \nabla \hat{v}_h$, where where $\hat{u}_h$ and $\hat{v}_h$ satisfy

$$\begin{array}{l} \hat{u}_h \in V_h(\omega^0), \quad \hat{u}_h|_{\Gamma_0} = u_h, \quad \hat{u}_h|_{\Gamma_1} = 0, \\ \hat{v}_h \in V_h(\omega^0), \quad \hat{v}_h|_{\Gamma_1} = v_h, \quad \hat{v}_h|_{\Gamma_0} = 0, \\ \forall w_h \in V_h(\omega^0) \text{ such that } w_h|_{\Gamma_1} = 0, \quad w_h|_{\Gamma_0} = 0, \quad \int_{\omega^0} \nabla \hat{u}_h \nabla w_h = \int_{\omega^0} \nabla \hat{v}_h \nabla w_h = 0. \end{array}$$

Denoting O the cone of the positive semi-definite matrices $Z \in \mathbb{R}^{N\times N}$ such that for $i = 1, \dots, N$, $\sum_{j=1}^{N} Z_{ij} = 0$, it is clear from the interpretations of S_{Γ^0,Γ^0}, S_{Γ^1,Γ^1} and S_{Γ^0,Γ^1} given above that the matrix counterpart of the operator $\mathbb{M}_h$ defined in (95) (96) is the operator $M : O \mapsto O$:

$$M(Z) = S_{\Gamma^0,\Gamma^0} - S_{\Gamma^0,\Gamma^1}\left(S_{\Gamma^1,\Gamma^1} + \begin{pmatrix} Z & 0 \\ 0 & Z \end{pmatrix}\right)^{-1} S^T_{\Gamma^0,\Gamma^1}. \qquad (100)$$

As a corollary to Theorem 10, we have the

Proposition 4 *For any $Z \in O$, the sequence $M^n(Z)$ converges geometrically to the unique fixed point T of M, and T is the matrix of the discrete Dirichlet-Neumann operator T_h^0 defined in (94) in the nodal basis of $V_h(\Gamma^0)$.*

5.3 Discrete Neumann problems

We consider now the discrete version of (26):

$$\text{find } u_h \in \mathcal{V}_h(\Omega^0) \text{ such that and for all } v_h \in \mathcal{V}_h(\Omega^0), \quad \int_{\Omega^0} \nabla u_h \cdot \nabla v_h = \frac{1}{3}\int_{\Gamma^\infty} gv. \qquad (101)$$

Once T_h^0 is computed, the approach proposed in §4 can be faithfully reproduced at the discrete level, namely,

1. the restrictions to ω^p (here we take $p = 2$) of the solutions to (101) with $g = 1$, $g = g^0$ the Haar mother wavelet given in § 4.4.1 and $g = g^\sigma$, $\sigma \in \mathcal{A}_n$, $1 \le n \le N$, see (63), are computed first, by reproducing at the discrete level the constructions given in § 4.3, and 4.4. These functions are respectively called $u_{F,h}$, u_h^0, and u_h^σ.

2. For a general function g, g is first written in the Haar wavelet basis, see (86), and the solution u_h to (101) is approximated by the truncated sum:

$$\alpha_F u_{F,h} + \alpha_0 u_h^0 + \sum_{n=1}^{N}\sum_{\sigma\in\mathcal{A}_n} \alpha_\sigma u_h^\sigma.$$

Since the discrete method is the transcription of the continuous one, we do not reproduce it. Rather, we choose to focus on the linear algebra used for the solutionof the discrete version of (54).

5.3.1 Solving (101) with $g = 1$

We obtain that $u_{F,h}|_{\omega^0}$ is the unique solution to : for $v \in \mathcal{V}_h(\omega^0)$,

$$\int_{\omega^0} \nabla u_{F,h} \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0 \left(u_{F,h} \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle = -\frac{1}{2}\sum_{i=1}^{2} \left\langle y_h, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle, \qquad (102)$$

where y_h satisfies: for all $v \in V_h(\omega^0)$,

$$\langle y_h, v\rangle = \int_{\omega^0} \nabla u_{F,h} \cdot \nabla v + \sum_{i=1}^{2} \left\langle T^0 \left(u_{F,h} \circ F_i\right), v|_{F_i(\Gamma^0)} \circ F_i \right\rangle + \frac{1}{2}\sum_{i=1}^{2} \left\langle y_h, v|_{F_i(\Gamma^0)} \circ F_i \right\rangle. \quad (103)$$

and

$$\langle y_h, 1\rangle = -\frac{1}{3}|\Gamma^\infty| = -2. \tag{104}$$

Let

$$U = \begin{pmatrix} 0 \\ U_{\Gamma^1} \\ U_I \end{pmatrix}$$

be the coordinates of $u_{F,h}|_{\omega^0}$ in the nodal basis (ϕ_i) introduced above. In matrix form, (102), (103) read

$$\begin{pmatrix} A_{\Gamma^0,\Gamma^0} & 0 & A_{\Gamma^0,I} \\ 0 & A_{\Gamma^1,\Gamma^1}+T' & A_{\Gamma^1,I} \\ A^T_{\Gamma^0,I} & A^T_{\Gamma^1,I} & A_{I,I} \end{pmatrix} \begin{pmatrix} 0 \\ U_{\Gamma^1} \\ U_I \end{pmatrix} = \begin{pmatrix} Y \\ -\frac{1}{2}Y' \\ 0 \end{pmatrix},$$

where Y is the vector representing y_h in the dual basis of $(\phi_i)_{i=1,\dots,N}$, and where

$$Y' = \begin{pmatrix} Y \\ Y \end{pmatrix}, \quad \text{and} \quad T' = \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix},$$

with T is the matrix of the discrete Dirichlet to Neumann operator.
After eliminating U_I, one obtains

$$\begin{pmatrix} S_{\Gamma^0,\Gamma^0} & S_{\Gamma^0,\Gamma^1} \\ S^T_{\Gamma^0,\Gamma^1} & S_{\Gamma^1,\Gamma^1}+T' \end{pmatrix} \begin{pmatrix} 0 \\ U_{\Gamma^1} \end{pmatrix} = \begin{pmatrix} Y \\ -\frac{1}{2}Y' \end{pmatrix}.$$

This implies the equation for Y:

$$Y + \frac{1}{2} S_{\Gamma^0,\Gamma^1} \left(S_{\Gamma^1,\Gamma^1} + \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right)^{-1} \begin{pmatrix} Y \\ Y \end{pmatrix} = 0.$$

which we can write

$$Y - BY = 0, \quad \text{where } B = -\frac{1}{2} S_{\Gamma^0,\Gamma^1} \left(S_{\Gamma^1,\Gamma^1} + \begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix} \right)^{-1} \begin{pmatrix} I_N \\ I_N \end{pmatrix} \tag{105}$$

is the discrete conterpart of B^0 introduced in (53) (54). It can be proved exactly as for Theorem 7 that Y is the unique solution to

$$\begin{aligned} (I_N - B)Y &= 0, \\ \sum_{i=1}^N Y_i &= -2. \end{aligned} \tag{106}$$

The problem (106) is equivalent to its least square form

$$((I_N - B^T)(I_N - B) + E)Y = -2 \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} \tag{107}$$

where E is the matrix in $R^{N\times N}$ whose entries are all 1. The matrix $((I_N - B^T)(I_N - B) + E)$ is symmetric and positive definite so (107) can be solved by means of the conjugate gradient method, which does not require assembly of B and B^T.

6 Numerical results

In the numerical tests, we have taken for Ω^0 a dilation by a factor π of the domain described in § 2. The mesh used for ω^0 is plotted on Figure 3. It has the property mentioned in § 5, which permits the construction of a self-similar mesh of Ω^0. On Figure 4, we have plotted two

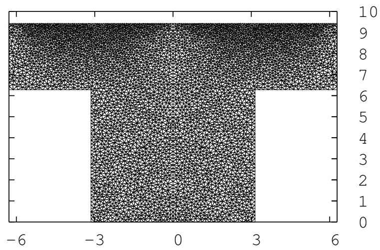

Figure 3: The mesh used for ω^0

views of $u_{F,h}$, computed by the method proposed in § 5.3.1, and of u_h^0,restricted to ω^2. On Figure 5, we have plotted $u_{F,h}$, u_h^0 and u_h^σ, $\sigma \in \mathcal{A}_1 \cup \mathcal{A}_2$. We see that the restrictions to ω^2 of the functions u^σ become smaller and smaller as n grows. To illustrate this better, we have plotted on Figure 6 the quantities and $\max_{\sigma\in\mathcal{A}_n} \|u_h^\sigma\|_{L^2(\omega^0)}$ for $n = 0, 1, 2, 3, 4$. We see clearly that the norm of $\|u_h^\sigma\|_{L^2(\omega^0)}$ decays exponentially with n, in agreement with Proposition 8. This shows that, for $g \in L^2(\Gamma^\infty$ such that $\|g\|_{L^2(\Gamma^\infty)} \approx 1$, the restriction to ω^0 of u solution to (26) to an accuracy of order 10^{-9} in L^2 norm can be computed by using expansion (88) up to $N = 3$ or $N = 4$ only.

On Figure 7 , we plot the solution to (26), with $g(x_1) = \text{sign}(-x_1)\cos(x_1)$ in ω^2, computed with expansion (88) up to $N = 4$ and $N = 2$. The two plots are visually identical. On Figure 8, we plot the quantities $\| \sum_{n=i}^4 \sum_{\sigma\in\mathcal{A}_n} \alpha^\sigma u_h^\sigma \|_{L^2(\omega^0)}$ and $\| \sum_{n=i}^4 \sum_{\sigma\in\mathcal{A}_n} \alpha^\sigma u_h^\sigma \|_{L^2(\omega^1)}$, for $i = 1, 2, 3$, where α^σ are the coefficients of the wavelet expansion of $g(x_1) = x_1 \cos(x_1)$. As expected, these quantities have a fast decay as i grows. This shows that carrying the expansion (88) up to $N = 3$ is enough to obtain $u|_{\omega^0}$ up to an error of order 10^{-15} in L^2 norm. To compute $u|_{\omega^1}$ with a comparable accuracy, it is necessary to carry the expansion further.

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References

[1] Y. Achdou, C. Sabot, and N. Tchou. Boundary value problems in some ramified domains with a fractal boundary: Analysis and numerical methods. part i: Diffusion and propagation problems. submitted.

[2] B. Adams, A. Smith, B. Strichartz, and A. Teplyaev. The spectrum of the laplacian on the pentagasket. In *Trends in Mathematics: Fractals in Graz 2001*, pages 1–24. Birkhauser, 2003.

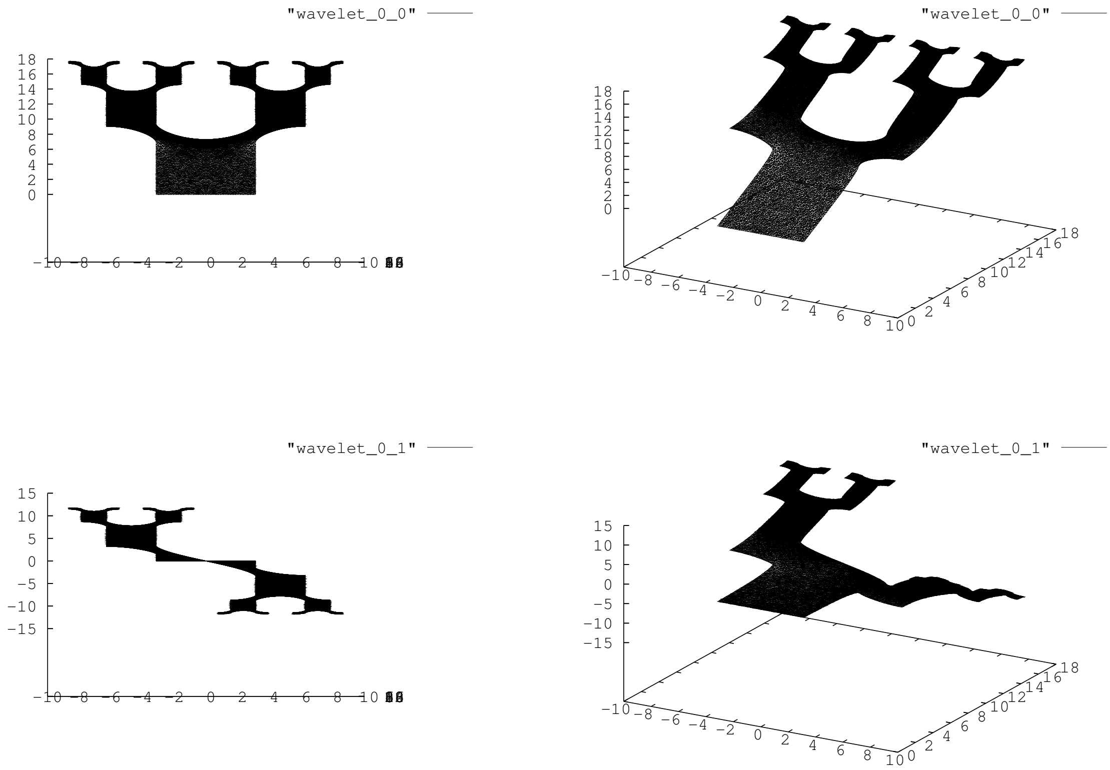

Figure 4: Two views of $u_{F,h}$ and of u_h^0, restricted to ω^2

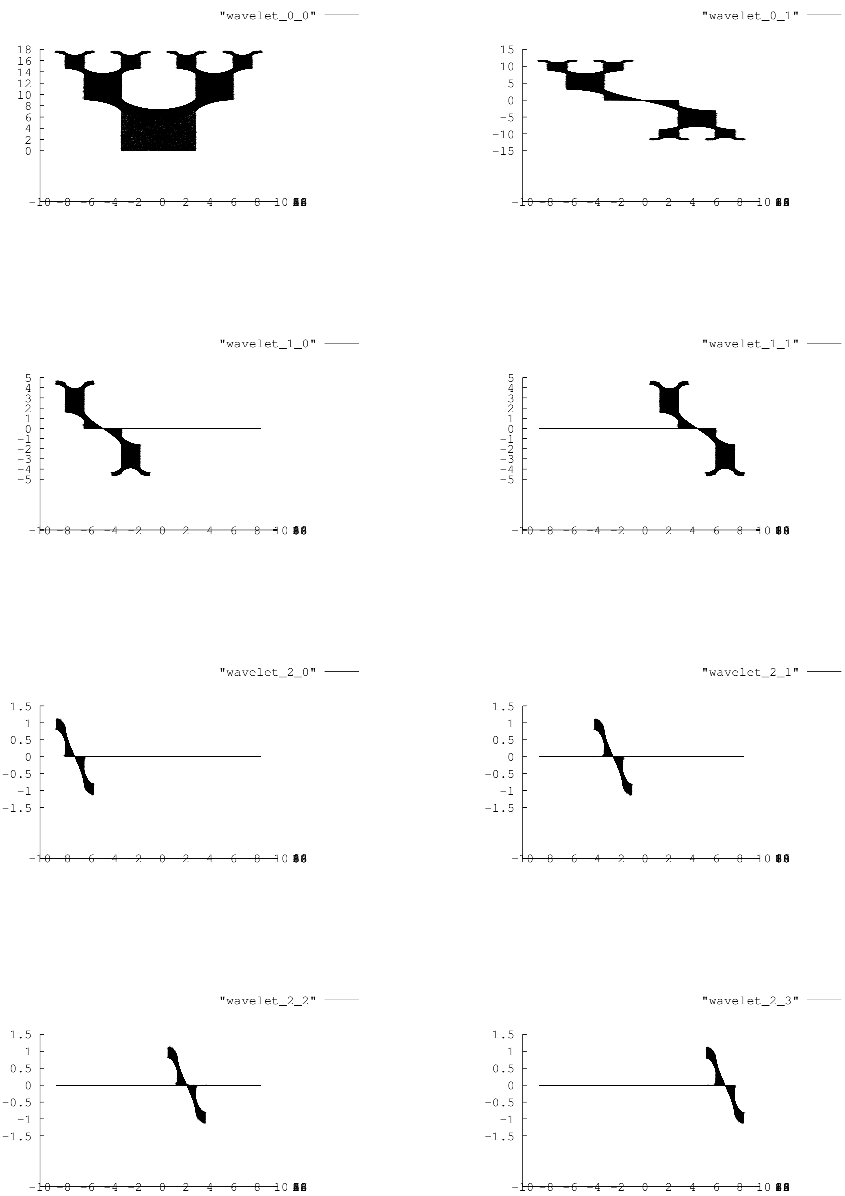

Figure 5: The restrictions to ω^2 of $u_{F,h}$, u_h^0, and u_h^σ, $\sigma \in \mathcal{A}_1 \cup \mathcal{A}_2$

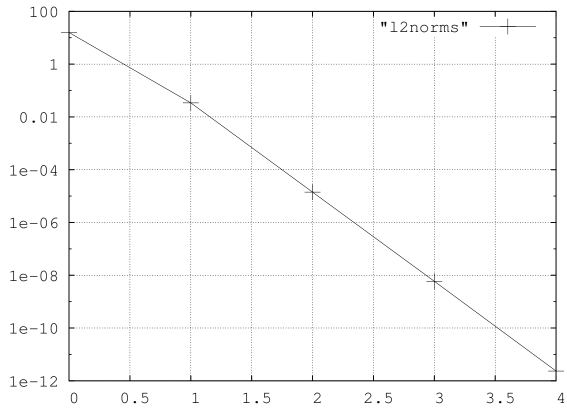

Figure 6: The quantity $\max_{\sigma\in\mathcal{A}_n} \|u_h^\sigma\|_{L^2(\omega^0)}$ vs. n. Note that $\|u_{F,h}\|_{L^2(\omega^0)} = 56.5862$.

[3] Robert A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[4] Haïm Brezis. *Analyse fonctionnelle.* Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree]. Masson, Paris, 1983. Théorie et applications. [Theory and applications].

[5] P. G. Ciarlet and J.-L. Lions, editors. *Handbook of numerical analysis. Vol. II.* Handbook of Numerical Analysis, II. North-Holland, Amsterdam, 1991. Finite element methods. Part 1.

[6] Michael Gibbons, Arjun Raj, and Robert S. Strichartz. The finite element method on the Sierpinski gasket. *Constr. Approx.*, 17(4):561–588, 2001.

[7] M. R. Lancia. A transmission problem with a fractal interface. *Z. Anal. Anwendungen*, 21(1):113–133, 2002.

[8] Maria Rosaria Lancia. Second order transmission problems across a fractal surface. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 27:191–213, 2003.

[9] U. Mosco and M. A. Vivaldi. Variational problems with fractal layers. *Rend. Accad. Naz. Sci. XL Mem. Mat. Appl. (5)*, 27:237–251, 2003.

[10] Umberto Mosco. Energy functionals on certain fractal structures. *J. Convex Anal.*, 9(2):581–600, 2002. Special issue on optimization (Montpellier, 2000).

[11] Richard Oberlin, Brian Street, and Robert S. Strichartz. Sampling on the Sierpinski gasket. *Experiment. Math.*, 12(4):403–418, 2003.

[12] R. Rammal. Spectrum of harmonic excitations on fractals. *J. Physique*, 45(2):191–206, 1984.

[13] Christophe Sabot. Electrical networks, symplectic reductions, and application to the renormalization map of self-similar lattices,. to appear in Proc. Symp. in Pure Math., "A Mandelbrot Jubilee".

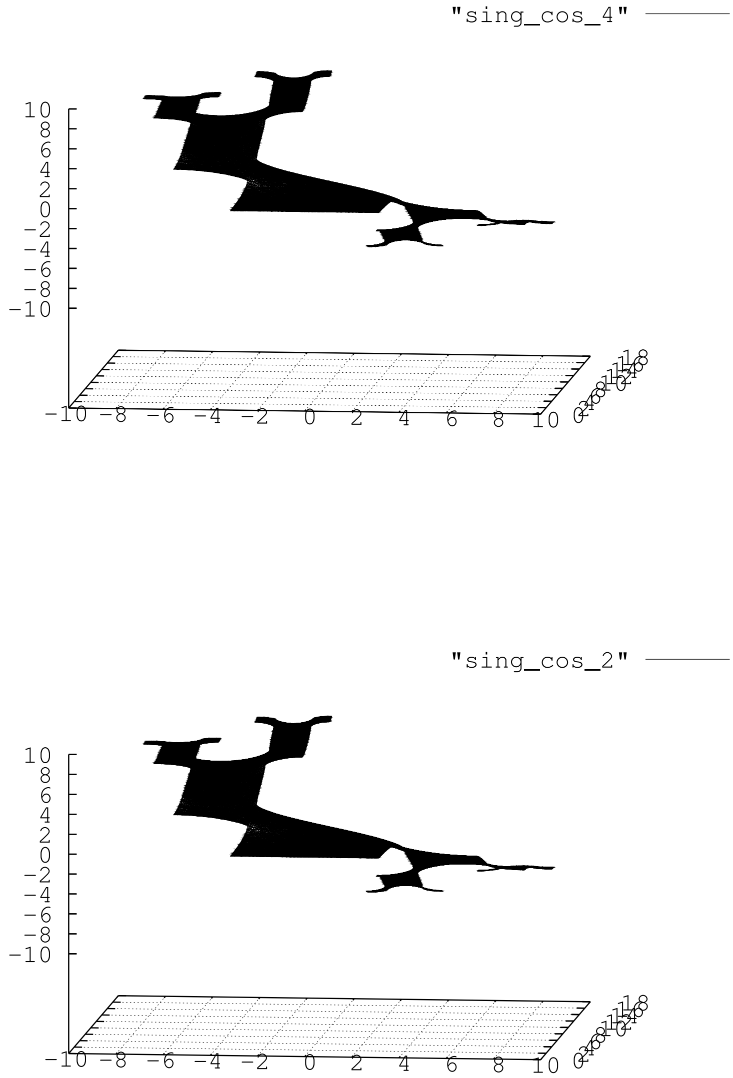

Figure 7: The restrictions to ω^2 of u solution to (26), with $g(x_1) = \mathrm{sign}(-x_1)\cos(x_1)$ in ω^2. The expansion (88) has been carried out up to $N = 4$(top), $N = 2$(bottom).

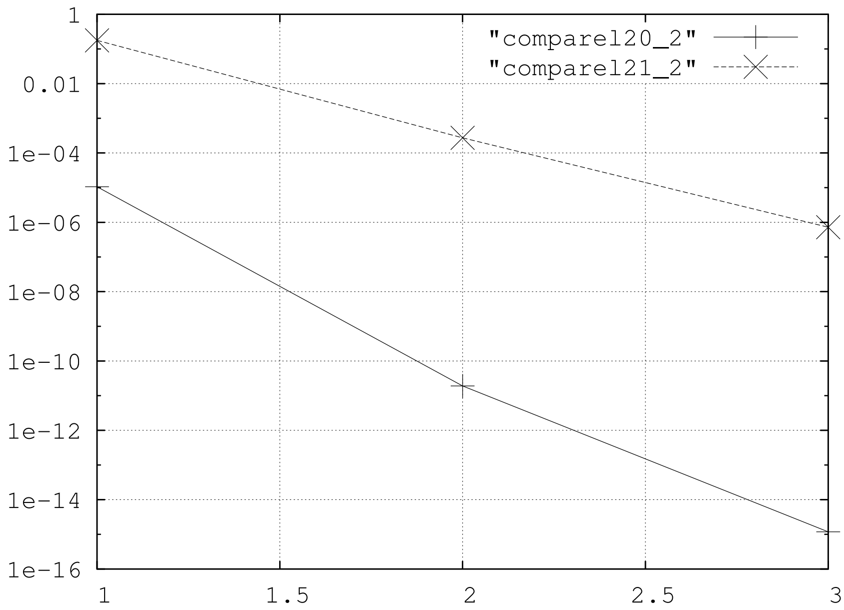

Figure 8: The quantities $\| \sum_{n=i}^{4} \sum_{\sigma \in \mathcal{A}_n} \alpha^\sigma u_h^\sigma \|_{L^2(\omega^0)}$ and $\| \sum_{n=i}^{4} \sum_{\sigma \in \mathcal{A}_n} \alpha^\sigma u_h^\sigma \|_{L^2(\omega^1)}$, for $i = 1, 2, 3$.

[14] Christophe Sabot. Spectral properties of self-similar lattices and iteration of rational maps. *Mém. Soc. Math. Fr. (N.S.)*, (92):vi+104, 2003.

[15] Alexander Teplyaev. Spectral analysis on infinite Sierpiński gaskets. *J. Funct. Anal.*, 159(2):537–567, 1998.